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# MATHEMATICAL
# ENGINEERING ANALYSIS

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# MATHEMATICAL ENGINEERING ANALYSIS

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*Rufus Oldenburger*, PH.D.
*Woodward Governor Company*

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*First Printing*

*To Eleanor*

# *Preface*

One of the most difficult tasks in industrial research is that of expressing physical situations in the form of equivalent mathematical relations. A good background in mathematics or engineering, or both, does not insure that the research engineer and scientist will have a facility for making the transition from physical engineering situations to the corresponding mathematics. This book was written to make the task of transition easier for the reader.

To operate at maximum efficiency the man in charge of an industrial research program today should have not only a thorough understanding of the experimental aspects of his problems but also should be able to set up his problems in mathematical form so that he can determine when something can be done more economically by mathematical computation than by experimentation, or vice versa. With the mathematics as guide he can better interpret experimental results and plan future physical studies. To this end he must have a knowledge of the theoretical physics concerned with the basic laws of engineering. All major fields of engineering should be represented because a problem will often involve several fields at the same time. It has been one of the author's objectives, in writing this book, to develop the basic laws of engineering from a minimum number of assumptions so that the reader can obtain a logical physical and mathematical picture of the fundamental concepts of engineering in common use. With this as a background, the various techniques for making simplifying assumptions in treating physical problems appear more reasonable. Such simplifying techniques are of inestimable importance, because if any physical problem were set up as precisely as possible in mathematical form, it would be too difficult to treat.

It is necessary for the industrial research man today to be able to set up physical problems, amenable to mathematical treatment, in a form that can be conveniently handled by mathematical methods. Experience renders invaluable assistance to him in this task. Since he will, of necessity, be constantly concerned with fields relatively new to him, he must fall back on the experience of others, on his knowledge of the basic laws of engineering, and on his ability to make simplifying assumptions. Several principles for making these assumptions, verified by experience, will be given throughout the chapters of the book, illustrated by many examples from the various domains of engineering.

This book is not intended to supplant the advanced courses in mathematics and the applied sciences. One engaged in research cannot learn too much about these subjects. We shall not be concerned here with the mathematical techniques for solving physical problems when they are expressed in mathematical form. For this purpose a thorough advanced training in mathematics is indicated. Fortunately, many problems do not require the most advanced mathematical methods for their solution to an accuracy required by practical considerations.

After many years of active research in pure mathematics, the author turned to applied mathematics of the type required by industry. Encountering numerous difficulties in setting up engineering problems in mathematical form, the author recognized the need for a course in the subject, and introduced one at the Illinois Institute of Technology, and also at De Paul University. The book is an outgrowth of these lectures and is intended to serve as a text in courses on engineering analysis and industrial physics and as a reference work for research directors in industry, for engineering analysts, for development engineers, and for applied scientists in various fields of learning.

The book assumes a knowledge of advanced calculus, especially those aspects concerned with line, surface, and volume integrals. Frequent use will be made of Green's and Stokes's theorems.

The author is indebted to various industries for their cooperation in combined mathematical-experimental investigations, in particular the Woodward Governor Company, which for the past eight years has carried on an active program of this type involving all of the major branches of engineering, and where many of the techniques mentioned in this book were developed and verified. The author owes much to his associate, Mr. Forrest Drake, chief engineer of the Woodward Governor Company, for seeing that advanced mathematics could be applied to the governor industry, and to Mr. Irl Martin, President of this company, as well as others for supporting this program before it had proved itself.

All or parts of the book were read by Professors Gordon Pall and Karl Menger of the mathematics department of the Illinois Institute of Technology, Professor Thomas J. Higgins of the electrical engineering department of the University of Wisconsin, Professor Eric Reissner of the mathematics department at the Massachusetts Institute of Technology, Professor Clark Millikan, director of the Guggenheim Aeronautical Laboratory at the California Institute of Technology, Professor Paul Lieber of the aeronautical department of the Brooklyn Polytechnic Institute, Professor Le Van Griffis, director of mechanics at the Illinois Institute of Technology, Professor George Irving Cohn of the electrical engineering department of the Illinois Institute of Technology, Dr. Gerhard Herzog, director of research of the Texas Company. Dr. Warren Wilson, President of the South Dakota School of Mines and formerly director of research of the Hydraulics Division of the Sundstrand Machine Tool Company, Dr. Eric Lype, thermodynamicist, and Mr. Kenneth Bisshopp, mechanical engineer, of the Armour Research Foundation, Mr. Forrest Drake and Mr. Urban Floor, engineer, of the Woodward Governor Company, and Mrs. Morris Kreeger.

To all of them the author owes his appreciation for helpful suggestions and constructive criticisms.

RUFUS OLDENBURGER

ROCKFORD, ILLINOIS

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# PART I

# MECHANICS OF RIGID BODIES

# CHAPTER

# 1

# *Fundamentals of the mechanics of particles and rigid bodies*

**1.1. Length, mass, and time.** In Part I of this book we shall be concerned with the setting up, in mathematical form, of physical problems involving rigid bodies.

Mass, length, and time can be considered to be the basic measurable quantities of mechanics. Geometry gives us an idea of the meaning of distance through the postulates which it satisfies. Distance in the physical world is defined by the operations we perform in measuring it. The same thing is true of other measurable entities.

Distance is a number associated with two points in space. It is therefore necessary to understand the meaning of a point before a definition of distance is attempted. Without bodies in space, points could not be recognized. We therefore begin with bodies in space which we know exist through our senses. If a body through some process becomes two or more bodies, we say that the given body has been *subdivided*, and the parts are *subbodies*. A body is said to occupy a *point* in space if we cannot subdivide it. By subdivision we do not mean an actual physical separation of a body into parts. We might subdivide a body by drawing a line on it, or by merely looking at it to discern subbodies. Thus, fundamentally, we arrive at the concept of a physical (not mathematical) point operationally by subdividing bodies. Basically, we recognize a point in space by having a body at that point.

Through our senses we observe that certain bodies are *rigid*. A body is *rigid* when the distance between any two of its points is unchanging, at least for practical purposes. Through our senses we also recognize when two bodies are geometrically *identical*.

A point is by definition zero-dimensional. A wire or rod is for practical purposes a one dimensional body, that is, we can treat the cross sections as points, so that we can think of the wire or rod as being in the form of a geometric curve.

We consider a collection of identical unchanging rods. The concept of identity of two rods implies that we can associate with every point on one rod a corresponding point on the other so that the first point bears the same relation to the first rod that the second point bears to the second rod. In Figure 1 we have pictured three identical rods. The ends labeled with the $A$'s correspond, and the same is true for the $B$'s. We suppose that these rods are hinged together to form a *ruler* as in Figure 2, the point $B_1$ of the first ruler being hinged to the end $A_2$ of the second, etc. We suppose that the rods of this ruler are free to move about the hinges except for the practical limitation that one rod cannot occupy the same space as another. In the general case we have a ruler made from $n$ such rods for an arbitrary $n$. It is convenient to treat *symmetric* rods such that two adjacent rods either touch at the hinge only, or they touch everywhere. Two rods *touch* at a point if for practical purposes they have this point in common.

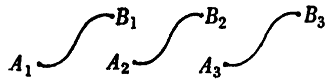

**Fig. 1. Identical Rods.**

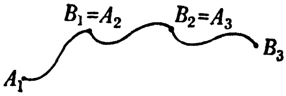

**Fig. 2. Hinged Ruler.**

We consider a rigid body as shown shaded in Figure 3. We select two points $A$ and $B$ on this body. We link $A$ and $B$ with a ruler hinged as in Figure 2. A rod being the unit of measure, if $A$ and $B$ are linked with $n$ rods the distance from $A$ to $B$ is $n$ or less units. For the example of Figure 3 we have $n = 4$. We suppose that the rods and given body are chosen so that we can scoop away part of the given body if necessary, so that for a linkage with $m$ rods, and $m$ properly chosen, no matter how we move the rods about on their hinges, keeping the $A$ and $B$ ends fixed, the rods never touch the body except at $A$ and $B$. If then we find a $q$ so that $A$ and $B$ can be linked with $(q + 1)$ rods, but not $q$, the *distance* from $A$ to $B$ is greater than $q$ units, but less than or equal to $(q + 1)$ units. Writing the

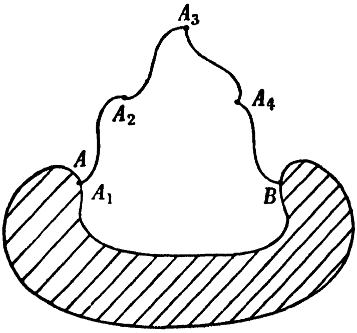

**Fig. 3. Points $A$ and $B$ connected by a hinged ruler.**

distance from $A$ to $B$ as $AB$, which is the same as the distance from $B$ to $A$, we have thus

$$q < AB \leqq q + 1. \tag{1.1}$$

We can form a ruler from subbodies of the given rods, and go through the above process to obtain in place of (1.1) a more accurate determination in terms of fractions of a unit. We arrive at a fraction of a unit by applying the above linking process to one of the original rods, using a ruler made from identical subbodies, and giving inequality relations of the type (1.1). The details will be left to the reader.

We let $A$, $B$, and $C$ designate three points on a one-dimensional affair such as a wire. The point $B$ is *between* $A$ and $C$ if the point $B$ is on a subbody of the wire joining $A$ and $C$. With the concept of distance physically defined we can construct a geometric model of the universe involving straight lines, coordinate systems, etc. Thus the wire is a segment of a *straight line* if for every three points $A$, $B$, and $C$, where $B$ is between $A$ and $C$ on the wire, we have

$$AB + BC = AC. \tag{1.2}$$

The concept of a straight line is an immediate extension of that of a straight line segment. With straight lines defined we can introduce a Cartesian *coordinate system* set up on a reference body for the purpose of studying motions of bodies in space. The problem of locating points is much simplified by assuming that *light rays travel in straight lines in empty space.*

*Angles* follow from the concept of distance.

Time is recognized by the observation of a body $B$ at a point $P$ being found at other points. We say then that *motion* has taken place. This is with reference to our coordinate system. If each body remained at the same position, we would say that there was no motion, and we could not recognize time.

We measure time through observation of cyclic motions. By a cyclic motion we mean one in which a body at a point $P$ leaves $P$ and describes a path $C$ in returning to $P$, then the body describes the same path $C$ again, and keeps doing this over and over, like the hand of a clock or the moving earth. We can say that the path $C$ is described in one unit of *time*. We can obtain fractions of this unit by the use of cyclic motions in which a path is described more than once while the path $C$ is described once. In practice the earth's motion forms the basis of a measurement of time, all clocks being referred to it.

With distance and time defined the calculus yields *velocity* and *acceleration.*

A body which, for practical purposes, is located at a single point is termed a *particle*. A *mathematical particle is a point with a mass at that point.* There may also be an electric charge or magnetic pole at the point.

If not otherwise specified, the term *particle* will refer to a mass-point. We let $m_1$ and $m_2$ be the masses of two particles $P_1$ and $P_2$ respectively, these masses being as yet undefined. We let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ denote the accelerations of $P_1$ and $P_2$ respectively. *Vectors are denoted in this book by symbols in boldface type.* We know from observation that if a body $B$ is far enough away from other bodies in the universe the motion of $B$ is negligibly affected by the presence of these other bodies. It is convenient to make the measurements of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ relative to a coordinate system attached to such a body $B$. From observation we know that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ will both be numerically large if $P_1$ and $P_2$ are sufficiently close. We therefore suppose that the particles $P_1$ and $P_2$ are close enough together so that each accelerates appreciably due to the presence of the other. We assume that $P_1$ and $P_2$ are sufficiently far removed from $B$ and the other bodies in the universe so that the influence of these bodies has a negligible effect on the motion of $P_1$ and $P_2$. We suppose that if the particles $P_1$ and $P_2$ carry electric or magnetic charges, these charges are small enough so that the motion of $P_1$ and $P_2$ is negligibly affected by such charges. We define the quotient $m_1/m_2$ by the relation

$$\frac{m_1}{m_2} = \left|\frac{\boldsymbol{a}_2}{\boldsymbol{a}_1}\right|, \tag{1.3}$$

where the vertical bars indicate absolute values. Taking the mass of one particle, say $P_1$, as a unit we can obtain the masses of all other particles, theoretically at least, by the relation (1.3).

The acceleration of a body $B$ due to the presence of a body $A$ is a measure of the amount the motion of $B$ is affected by $A$.

In the above discussion we have not concerned ourselves with relativistic effects. *Observation shows that $m_1/m_2$ is then a constant.*

We now consider a set of particles $\{P_i\}$ where $i$ ranges over 1, 2, . . . , $n$, and $m_i$ denotes the mass of $P_i$. Let two particles $P_i$ and $P_j$ of this set have the accelerations $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$ when far removed from other bodies. By "far removed" we mean that we should, to be rigorous, have in mind a mathematical limit process where we let the distance from $P_i$ and $P_j$ to other bodies increase beyond all bounds. The following law follows from observation.

*Law* **1.1.** *Between the m's and* ***a****'s we have the relation*

$$\frac{m_i}{m_j} = \left|\frac{\boldsymbol{a}_j}{\boldsymbol{a}_i}\right| \tag{1.4}$$

*valid for all values of i and j.*

The reader can easily see that the Law 1.1 is a consequence of Newton's laws of motion to be given later. With the present point of view Newton's second law of motion will follow from the concept of mass as defined here.

Law 1.1 merely states that if we have determined the masses of particles in terms of a given one, these masses hold when any two of these particles are compared in far away space.

We have the following law.

***Law* 1.2.** *For the particles $P_i$ and $P_j$ the accelerations are directed along the same line and toward each other.*

When we have defined force it will appear that Law 1.2 and the definition of mass above yield *Newton's third law of motion* as applied to these particles, namely, that *to every action there is an equal and opposite reaction*. This will mean that the force at $P_i$ is numerically equal and oppositely directed to the force on $P_j$.

Law 1.2 follows from observation.

**1.2. Force.** We recognize forces through our muscles. With our point of view *Newton's second law of motion*, which follows, *is not a law, but a definition of force.*

***Newton's second law of motion.*** *The force $\boldsymbol{F}$ acting on a particle of mass $m$ and acceleration $\boldsymbol{a}$ is by definition such that*

$$\boldsymbol{F} = m\boldsymbol{a}. \tag{1.5}$$

Newton actually states his law in the more general form

$$\boldsymbol{F} = \frac{d(m\boldsymbol{v})}{dt}, \tag{1.6}$$

which says that the force $\boldsymbol{F}$ applied to the particle equals the rate of change of *momentum* $m\boldsymbol{v}$ of this particle, the velocity of the particle being designated by $\boldsymbol{v}$. The symbol $t$ denotes time.

For usual engineering purposes the mass $m$ is constant, so that (1.6) reduces to (1.5). From the Einstein formula

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{|\boldsymbol{v}|}{c}\right)^2}} \tag{1.7}$$

for the rest mass $m_0$ of the particle and speed of light $c$, it is clear that the assumption of the constancy of the mass is justified unless the particle is traveling almost as fast as the speed of light, relative of course to the given coordinate system.

Forces not expressed in the form (1.5) are obtained experimentally. Consider, for example, an ordinary helical spring. By keeping one end of the spring fixed and attaching a body with known mass to the other end, measurements of the acceleration of the body for different lengths of the spring and the application of (1.5) enable us to associate a force with each length of the spring, making allowance if necessary for the mass of the spring itself. Knowing the law connecting such a force with the length of the spring, we need only measure the length of the spring to get the force associated with it, experience telling us that we always get the same force for each length within elastic limits. Thus forces can be obtained without measuring masses and accelerations in each experiment.

In practice masses are usually measured by weighing bodies, thus introducing the earth, as well as forces into the picture. This involves moments to be explained later. We can similarly use moments to measure the force due to contraction of a spring.

Since for $m \neq 0$ we have $a = 0$ if and only if $F = 0$, equation (1.5) implies that a particle with constant mass $m$ remains at rest or travels with constant nonzero velocity unless acted on by some force. This is *Newton's first law of motion* applied to this particle.

**1.3. Gravitation.** For particles with masses $m_1$ and $m_2$, separated by a distance $x$, the Newtonian law of gravitation is as follows.

***Law* 1.3. Newton's law of gravitation.** *For particles of mass $m_1$ and $m_2$ a distance $r_{12}$ apart the gravitational force $F$ attracting $m_1$ to $m_2$ has the magnitude $|F|$ given by*

$$|F| = K\frac{m_1m_2}{r_{12}^2} \tag{1.8}$$

*for a constant $K$, and $F$ is directed toward $m_2$ along the line through these particles.*

By "$F$ attracting $m_1$ to $m_2$" we mean a force $F$ at the first particle due to the presence of the second. Newton's law of gravitation follows from the study of the motions of the heavenly bodies. This law says that $|F|$ is proportional to the product of the masses, and varies inversely as the square of the distance between them.

Although it is actually impossible to measure the gravitational force of attraction between two particles, Newton's law of gravitation has been verified indirectly by matching computations based on this law against measurements of the gravitational force between conveniently chosen bodies. This is readily done for two homogeneous spheres, since it can be shown that (1.8) holds when $r_{12}$ is the distance between the centers of these spheres.

Except for small discrepancies, which need the theory of relativity for their explanation, Newton's gravitational law fits the movements of planetary bodies. At the other extreme this law does not agree with the force of attraction found to exist between atomic particles. For atomic particles the law of attraction seems to be more like (1.8) with a high power like the seventh on $r_{12}$ so that the force is very great when the particles are close together, and drops off rapidly as they separate.

The gravitational force of attraction between two one-pound weights one foot apart at sea level is about $32 \cdot 10^{-12}$ pounds. From this it is seen that gravitational forces between bodies on the earth's surface, arising in engineering considerations, are usually small compared to the other forces acting on these bodies, and may therefore be neglected. An exception is found in the geophysical prospecting for oil. Different kinds of oil-bearing strata are of different densities, so that the attraction of such a mass for the bob of a pendulum will vary with the strata.

Since anything a man can build has geometric dimensions small compared to the earth, in treating the attraction between the earth and a piece of equipment we may suppose that the equipment is a particle concentrated, for example, at the center of gravity to be defined later. To analyze the motion of a projectile, like the German V-2 rocket bomb, the relation (1.8) can be used, $r_{12}$ being the distance from the center of the earth to the center of gravity of the bomb. Since the earth is not a perfect homogeneous sphere there will be some discrepancy in the results which can be corrected in the formula for the acceleration of the bomb due to the force of gravity. This formula is

$$|a| = \frac{km}{r^2} \tag{1.9}$$

for a real number $k$, the acceleration $a$ of the bomb, the mass $m$ of the earth, and the distance $r$ from the center of the earth to the center of gravity of the bomb. Relation (1.9) is an immediate consequence of (1.5) and (1.8).

If a body $B$ remains at practically the same altitude relative to sea level, the corresponding distance $r$ in (1.9) is practically constant. By (1.9) this means that, regardless of the mass $m_B$ of $B$, the acceleration of $B$ is constant when $B$ is falling subject to the force of gravitational attraction of the earth only. For practical purposes it is accurate enough to take the magnitude $g$ of this acceleration as 32 feet per second per second (32 ft per sec$^2$). This figure has been obtained, of course, by physical experiments. We have also 386 inches per second per second (in per sec$^2$) for g.

That the gravitational force of attraction of the earth on $B$ is directed toward the center of the earth follows at once from symmetry. We do not need equation (1.9) to show that the acceleration of $B$ is independent of $m_B$. Using relation (1.8) for the force of attraction between $B$ and an element of mass of the earth, we can write an integral representing the force $F$ of attraction of the earth on $B$, where the integration is over the region occupied by the earth. The mass $m_B$ is a factor of the integrand function, and can be taken out of the integral. It follows that $|F|$ is $gm_B$ for a constant $g$, whence the acceleration of $B$ is independent of $m_B$.

We can introduce mass through (1.8), thus obtaining *gravitational mass* in contrast to *inertial mass* defined in Law 1.1. No disagreement has ever been found between these two methods of introducing mass. In the theory of relativity these are assumed to be the same.

**1.4. Superposition of forces.** Sometimes several forces are acting on a particle at the same time. For this reason we shall need the law to follow.

*Law* **1.4.** *The forces acting on particles $P_1$ and $P_2$ due to these particles is unaffected by the presence of other particles.*

Thus, starting with $P_1$ and $P_2$ we can introduce a third particle $P_3$ without affecting the forces already acting on $P_1$ and $P_2$. In addition to these forces we have forces acting on $P_1$ and $P_2$ due to the presence of $P_3$, and of course, there will also be forces acting on $P_3$ due to the presence of $P_1$ and $P_2$.

Law 1.4 holds if the forces due to electric and magnetic charges are introduced. These charges may be at the particles themselves.

Thus far we have treated individual particles. It is ordinarily convenient to think of bodies as continuous mass distributions even though this is not the case. Remarkable agreement between theory and experiment is obtained by this assumption. Replacing a body by a set of particles we can by the methods of integration in calculus extend results for individual particles to continuous mass distributions. This will be illustrated in the discussion of the center of gravity to follow in a later section.

Law 1.4 holds when particles are replaced by continuous mass distributions. This follows because, as noted above, we can approximate continuous mass distributions by finite numbers of particles. In particular, *forces on rigid bodies are unaffected by the introduction of other forces.*

The following stronger form of Law 1.4 holds.

*Law* **1.5.** *The force acting on a particle A due to the presence of other particles is the sum of the forces acting on A due to the presence of the other particles individually.*

That velocities, accelerations, and forces are vectors follows immediately from the definition of velocity and acceleration in terms of derivatives and the proportionality of a force to an acceleration.

**1.5. Dimensions.** In checking formulas it is convenient to replace numbers representing the measure of physical entities by symbols indicating the entities measured. Thus we denote force, mass, length, and time by the symbols $F$, $M$, $L$, and $T$, respectively. Then in dimensional form we have

$$F = \frac{ML}{T^2}. \tag{1.10}$$

Arithmetic multipliers not denoting the measure of a physical quantity are disregarded in replacing a formula by its dimensional equivalent. Dimensional analysis is often used for formulas that are polynomials in the measured quantities.

Velocity is dimensionally $L/T$, and acceleration $L/T^2$, whence (1.5) yields (1.10).

Dimensionally (1.9) is

$$\frac{L}{T^2} = \frac{kM}{L^2}, \tag{1.11}$$

whence $k$ is dimensionally given by

$$k = \frac{L^3}{T^2M}. \tag{1.12}$$

If $x$ is the displacement of a linear spring from equilibrium, the spring force $F$ is given by $Kx$ (Hooke's law), whence $K$ is dimensionally

$$K = \frac{M}{T^2}. \tag{1.13}$$

Engineers often prefer the dimension of force to mass. In this case the dimensionality of $K$ is $F/L$. One advantage of the use of dimensions lies in changing units of measure. Thus if $K = 200$ when pounds and inches are used, we can get $K$ for pounds and feet by replacing $L$ in $F/L$ by $L/12$ so that $F/L$ becomes $12F/L$, whence $K = 2{,}400$ in pounds per foot. With 16 ounces in a pound, if we use ounces and feet we must replace $F/L$ by $16F/L$, where in the latter fraction $F$ is in ounces and $L$ in feet, whence $K$ goes from 2,400 to 38,400 ounces per foot. Here we are using $F$ and $L$ to mean force and length in given units of measure.

In setting up equations of motion for physical systems we add or subtract terms only if they are dimensionally the same. For example, we add pounds to pounds, or seconds to seconds. This fact is of value in checking the equations.

Dimensions are also of use in finding new formulas, as will be shown in Sections 2.20–2.23 and sections of other parts of the book.

## 1.6. Center of gravity.

The reader is assumed to be familiar with the concept of *resultant* of a finite number of forces, obtained by adding these forces according to the parallelogram law. We shall consider a continuous mass distribution that will require integration of real functions representing force components.

We shall show how the concept of *center of mass*, or what for practical purposes in engineering is the same thing, *center of gravity*, simplifies the study of rigid bodies. Although *center of mass* suggests a point that does not depend on a force of gravity acting on the body, but rather depends on the body only, we shall adopt the usual engineering language of *center of gravity.*

We consider a Cartesian coordinate system in three-dimensional Euclidean space and a rigid body $B$ in this space as shown in Figure 4. We choose a point $P_i(x_i, y_i, z_i)$ on or within the body $B$, and construct an element of volume containing the point $P_i$, for convenience a rectangular parallelepiped with sides parallel to the coordinate axes. The length of these sides is taken to be $\Delta x_i$, $\Delta y_i$, and $\Delta z_i$ respectively, and the mass of the body contained within the element as $\Delta m_i$. We construct a sequence of such parallelepipeds with $i = 1, 2, \ldots, n$ having the property that these parallelepipeds do not overlap, and each point in the body $B$ lies on or within at least one such parallelepiped. We let $M$ designate the mass of the body $B$, and let $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ denote the *center of gravity* of $B$.

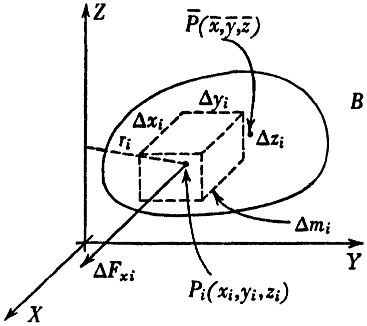

**Fig. 4. Rigid body $B$ with center of gravity $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ and element of mass $\Delta m_i$.**

Consider the lengths of the chords joining points of the same parallelepiped. For the given parallelepipeds these lengths have a maximum value $\delta$. By definition $\bar{x}$ is such that the equation

$$\bar{x}M = \lim_{\delta \to 0} \sum_{i=1}^{n} x_i \Delta m_i \tag{1.14}$$

is satisfied, where the limit is taken over sequences of parallelepipeds as $\delta \to 0$. In standard notation (1.14) can be written as

$$\bar{x}M = \iiint_B x\,dm, \tag{1.15}$$

the integration being over the body $B$. Similarly,

$$\bar{y}M = \iiint_B y\,dm, \qquad \bar{z}M = \iiint_B z\,dm, \tag{1.16}$$

so that the point $\bar{P}$ is uniquely determined by (1.15) and (1.16), provided of course that the indicated integrations can be carried out.

To clarify the integrations of (1.15) and (1.16) we shall introduce the concept of *density at a point.* Let $\Delta V$ denote the volume of an element containing the point, and let $\Delta m$ designate the mass of this element. The *density* of the body at the point in question is $\rho$, where

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}; \tag{1.17}$$

in this limit the maximum chord joining points of the element goes to zero. It follows from (1.17) that if the density $\rho$ of a body is the same at each point in the body, the mass of this body is the product of its volume by the density $\rho$.

Let us assume that the density varies continuously over the body $B$, whence the density takes on a maximum and minimum value for each element. We let $\rho_{iM}$ be the maximum value of the density at the points of the element of Figure 4, and $\rho_{im}$ be the minimum. If the density at each point in the element is replaced by the maximum $\rho_{iM}$, the mass of the element remains the same or is increased. A similar remark holds for $\rho_{im}$ with the word *increased* replaced by *decreased.* We therefore have

$$\rho_{im}\Delta x_i \Delta y_i \Delta z_i \leqq \Delta m_i \leqq \rho_{iM}\Delta x_i \Delta y_i \Delta z_i. \tag{1.18}$$

It follows that

$$\Delta m_i = \rho_i \Delta x_i \Delta y_i \Delta z_i \tag{1.19}$$

for some number $\rho_i$ between $\rho_{im}$ and $\rho_{iM}$. Since the density is assumed to vary continuously with a point in $B$, the value $\rho_i$ is taken on by some point in the element; for simplicity let us say that it is $P_i$. We can now write (1.14) as

$$\bar{x}M = \lim_{\delta \to 0} \sum_{i=1}^{n} x_i \rho_i \Delta x_i \Delta y_i \Delta z_i, \tag{1.20}$$

whence by the methods of the calculus we have

$$\bar{x}M = \iiint_B x\rho\, dx\, dy\, dz \tag{1.21}$$

for (1.15), the limits of integration being obtained from the body $B$ as is usual in the calculus.

The properties of continuous functions used here enter implicitly into virtually all extensions of physical laws from the finite to the continuous case.

Let us consider the body $B$ at a particular instant. The point $P_i$ of the body $B$ has an instantaneous $x$-component of acceleration represented by a real function $a_{xi}$, which by calculus is $d^2x_i/dt^2$, time being denoted by $t$. We define an element of force by the function $\Delta F_{xi}$, for which

$$\Delta F_{xi} = a_{xi}\Delta m_i. \tag{1.22}$$

Integrating we have the following relation for the function $F_x$ representing the $x$-component of the force acting on the body $B$, the double prime designating second derivative with respect to t:

$$F_x = \iiint_B x''dm. \tag{1.23}$$

The following theorem relates the motion of a rigid body $B$ to that of a particle.

**Theorem 1.1.** *The center of gravity of a rigid body B under the influence of a system of forces moves as if the mass of B were concentrated at the center of gravity and all of the forces were acting at this point.*

The $x$-coordinate $\bar{x}$ of the center of gravity of $B$ may vary with the time $t$, and so may the $x$-coordinate of a point anywhere in $B$. The mass $\Delta m$ of an element is assumed not to vary with time, the same being then true of the total mass $M$ of $B$. We can obtain

$$\bar{x}''M = \iiint_B x''dm \tag{1.24}$$

by differentiating both sides of (1.15) twice, provided that the indicated differentiations can be performed. By (1.23) and (1.24)

$$F_x = \bar{x}''M \tag{1.25}$$

as was to be shown. As far as the motion of the center of gravity is concerned a rigid body thus behaves like a particle.

By Theorem 1.1 the study of the motion of a rigid body reduces to that of the center of gravity and motion of the body about the center of gravity. The motion of a body about a fixed point, which may or may not be the center of gravity, can be thought of as a rotation about an axis through this point, the axis itself being possibly in motion. The rotation of a body about an axis is of particular importance for engineering purposes. This will be treated later.

**1.7. Internal forces.** A force which acts on a physical system but depends on something outside of this system is called an *external force*. Forces which depend only on the interaction of parts of the system are termed *internal*. These concepts are important for the theorem which follows.

**Theorem 1.2.** *For a system of particles the sum of the internal forces vanishes.*

Theorem 1.2 follows immediately from Laws 1.1, 1.2, and 1.5. By integration it extends to a continuous distribution of matter. It follows that the center of gravity of a rigid body moves as though the total external force were applied at the center of mass with the mass of the body concentrated at this point.

**1.8. Momentum.** The momentum of a system of particles is by definition the sum of the momenta of these particles (see Section 1.2). By integration this extends immediately to continuous distributions of mass.

By (1.6) and Theorem 1.2 the momentum of a system of particles is constant if there is no external force acting on this system. It follows that if for practical purposes the sum of the external forces acting on a system is small, we may take the momentum of the system constant. This we have done in a later section on the collision of two bodies.

**1.9. Energy.** The concept of energy springs from the idea of *work* to which it is equivalent. Through getting tired we feel that something in us disappears when we do "work" against an opposing force. Aside from this, energy and work are concepts which arise naturally in the mathematical study of bodies in motion.

Let us suppose that a particle of mass $m$ moves along an $x$-axis under the influence of an applied force represented by a real continuous function $F$ of $x$. By definition the work done by the force, represented by $F$, in moving the particle from $x = a$ to $x = x$ is given by the integral

$$W = \int_a^x F dx, \tag{1.26}$$

whence differentiation yields

$$\frac{dW}{dx} = F. \tag{1.27}$$

By (1.5)

$$dW = mx''dx. \tag{1.28}$$

Here the primes denote differentiation with respect to time. Integration of (1.28) yields

$$W = \tfrac{1}{2}m(x')^2 + C. \tag{1.29}$$

The integral in (1.26) will still exist if $F$ has a finite number of discontinuities and is bounded. We define the *kinetic energy* $T$ of the particle to be such that

$$T = \tfrac{1}{2}m(x')^2. \tag{1.30}$$

It follows that the work done in going from $x = b$ to $x = c$ is the kinetic energy at $x = c$ minus the kinetic energy at $x = b$.

If the work done is positive we say it was done by the force and if negative the work was "done on" the force.

These considerations generalize to the motion of a particle in space, where the work done in describing a path $C$ is the integral

$$\int_C \mathbf{F} \mid \cos \theta ds$$

for the magnitude $|F|$ of the force $F$, an angle $\theta$ between the vector $F$ and a tangent ray in the direction of motion, and arc length $s$ on $C$. The work done in describing $C$ can be written in the vector form

$$\int_C F \cdot ds.$$

In general, by definition,

$$T = \tfrac{1}{2}m|v|^2 \tag{1.31}$$

for the velocity $v$ of the particle. This relation extends to a system of particles, where *the kinetic energy of the system is the sum of the kinetic energies of the particles.* Integration yields the kinetic energy of a continuous mass distribution.

We can define other forms of energy such as heat, and shall accept without proof the following law.

***Law of conservation of energy.*** *Energy can neither be created nor destroyed.*

Thus energy possessed by a system independently of other systems either transforms into a different form or to its equivalent mass. The principle of the equivalence of mass and energy is of practical importance in the case of the atom bomb.

Energy may be received by a system or delivered to other systems.

We may think of the law of the conservation of energy as due to the way we define energy.

Energy is recognized through its equivalent *work.* Energy manifests itself in mechanical, electrical, magnetic, thermal, chemical, and other forms. A more extensive treatment of the subject will be given in Part III on thermodynamics.

It is sometimes convenient to think of any kinetic energy change of a system as due to a force. The same statement is true for other forms of energy possessed by a system. We can clearly replace (1.27) by

$$F = \frac{dT}{dx}, \tag{1.32}$$

which can be thought of as defining force. A more convenient form for our purpose is

$$Fx' = T' \tag{1.33}$$

for the time derivative $x'$ representing the velocity $v$ of the particle, and time derivative $T'$ of $T$.

**1.10. Collision of two bodies.** The study of force naturally leads us to consider the theory of collision of molecules. We treat the more general case of the collision of two bodies which are not distorted too much because of the collision, so that as far as computations of momenta are concerned we may suppose that they are rigid, and concentrated at their centers of mass.

We let $m_1$ and $m_2$ be the masses of two bodies. We suppose that the motion takes place along a straight line. We let $v_1$ and $v_2$ be real functions representing the respective initial velocities of the two bodies. We let $V_1$ and $V_2$ represent their respective final velocities after impact. Since the momentum of the system of two bodies is constant we have the equation

$$m_1v_1 + m_2v_2 = m_1V_1 + m_2V_2. \tag{1.34}$$

We need another equation to determine the final motion.

We shall suppose, to begin with, that the kinetic energy of the system is unchanged. This means that no energy goes into heat or some other form. We have thus

$$\tfrac{1}{2}(m_1v_1^2 + m_2v_2^2) = \tfrac{1}{2}(m_1V_1^2 + m_2V_2^2) \tag{1.35}$$

as the conservation of energy equation.

We write (1.34) in the form

$$m_1(v_1 - V_1) = m_2(V_2 - v_2). \tag{1.36}$$

From this it is clear that either the initial and final velocities of both bodies are equal (no collision at all), or

$$v_1 \neq V_1, v_2 \neq V_2. \tag{1.37}$$

We assume (1.37). We write (1.35) as

$$m_1(v_1^2 - V_1^2) = m_2(V_2^2 - v_2^2). \tag{1.38}$$

By (1.36), (1.37), and (1.38)

$$v_1 + V_1 = v_2 + V_2. \tag{1.39}$$

The equations (1.36) and (1.39) determine $V_1$ and $V_2$ in terms of $m_1$, $m_2$, $v_1$, and $v_2$.

The equations (1.36) and (1.39) hold for perfectly elastic collisions, which is the case for ordinary molecular impacts.

When bodies are not perfectly elastic, as is usually the case in engineering, it is convenient to replace (1.39) by

$$V_1 - V_2 = e(v_2 - v_1). \tag{1.40}$$

When $e = 1$ equation (1.40) reduces to (1.39). At the other extreme when $e = 0$ we have $V_1 = V_2$. This is the case of completely inelastic collision when the bodies adhere after collision. In practice we never have $e = 0$, although it is nearly zero for putty or lead in the case of ordinary collisions. In any event the momentum equation (1.34) holds.

Theory and experiment show (1.40) to hold for normal head-on collisions, and that $e$ depends mainly on the materials of which the bodies are made. Values of $e$ for such a material as steel can be found in engineering tables. For steel a common value of $e$ is 0.55.

**1.11. Velocity, position, and inertial forces.** If a rigid body moves through a homogeneous fluid medium, such as a piston in a lubricated cylinder or an airplane in air, the body encounters resistance. This is due to the momentum transferred to the fluid because of impact with its particles. From physical considerations we would expect the body to encounter more fluid particles as the speed is increased, and that as the speed $v$ of the body is increased, more kinetic energy is transferred to the fluid particles, on the average. We would therefore expect the rate $T'$ of this kinetic energy to increase as a power of the speed, such as the square of the speed, so that

$$T' = cv^2, \tag{1.41}$$

where we have supposed, for simplicity, that the motion of the body is one-dimensional. By (1.33) we have (consider the body to be drifting)

$$|\boldsymbol{F}| = cv \tag{1.42}$$

for the *velocity force* $\boldsymbol{F}$. The force $\boldsymbol{F}$ is generally called a *damping force*. The constant $c$ is the damping coefficient. It has the dimensionality $M/T$, and $FT/L$.

In any case (whether or not the motion is linear) the damping force $\boldsymbol{F}$ will be represented by an increasing function $F(v)$ of $v$ where

$$F(0) = 0. \tag{1.43}$$

Experiments show that we can use

$$F(v) = cv^n, \tag{1.44}$$

where $n$ depends on the range of speeds being studied. For high-speed projectiles the exponent $n$ may be as much as 3, and is often taken equal to fractional values like 2.5. At the speeds that ordinarily arise in engineering $n = 1$ is satisfactory for many applications.

If the medium is changing, or the attitude of the body relative to the direction of motion is varying, we can account for this by changing $c$.

The above remarks hold also for nonrigid bodies provided the changes in shape are not too large. Also, the medium may be a collection of dust particles or even a solid medium which breaks to allow passage of the given body.

The damping may be due to friction within the body itself. The considerations above indicate that this would increase with the relative motion of the molecules in the body.

We have shown that damping arises when a body moves through a medium, this damping being due to molecular collisions, and being thus a force due to contact. Understanding that a force given by a function of the velocity of a point in a body is a *velocity force*, it will appear from Part II on electricity and magnetism that such a force can also arise due to "action at a distance." This force of Part II also depends on position. Specifically, an electric charge moving with velocity $\boldsymbol{v}$ exerts a force (with magnitude proportional to $|\boldsymbol{v}|$) on other electric or magnetic charges, so that if the body being studied is made of magnetic material or conducts electric current a force is exerted on it *due to currents* in other bodies.

The *forces due to collisions and currents are the physical velocity forces which arise in the analysis of motions.* By a *physical force* we mean one that exists because of the presence of matter in space, in contrast to forces which can be applied by a living being. Because of the nature of the human mind, forces applied by muscular effort can be almost anything in mathematical form, such as rather arbitrary functions of time.

In setting up the equations of motion of a physical system it is not necessary in general to work with the coordinates of each point in the system, in fact, if we did so the mathematics would be too cumbersome. It is necessary only to know the positions of certain *reference points* from which the positions of all other points can be determined. We use "point" here to mean position occupied by a particle of the system. If the body is rigid it is a simple matter to locate a point with reference to the rest of the body. In practice we focus our attention on a small portion of the body, which might, for example, be marked with a pencil.

It is similarly a simple matter to follow the motion of "particles" in a deformable solid body. For liquids it is not so easy. In a practical case we could, for example, suspend small solid particles in the fluid, assuming that this does not affect the fluid. As will be seen in Part V on fluid mechanics, it is generally necessary to work with each point in the fluid to study its motion.

The position of each point in a rigid body is determined when the positions of three points in the body are known. The position of each point is also known when the position of the center of mass and a set of fixed axes on the body with origin at the center of mass are known relative to the given coordinate system.

In studying machinery one coordinate will often suffice for each body in the given system. Thus the position of a piston in a cylinder and of a wheel on an axis are determined by the value of one variable.

Forces which depend on distance only will be called *position forces.* Gravitational forces on given bodies depend on position only and are position forces. As will be seen from Part II the "static" electric and magnetic forces of attraction and repulsion between given charges and poles are also position forces.

When a spring is extended or contracted, or for that matter an elastic body is deformed, the equilibrium between the molecular forces is disturbed. We would expect the rearrangement of the molecules to give a different resultant force dependent on the displacement because the gravitational, electric, and magnetic forces between the molecules would be changed. This is what happens.

For an elastic body a given set of forces will produce a certain shape. If now the forces are varied and then the original applied forces are used, the body will come back, for practical purposes, to its original shape. If this did not happen we would say that the body was not elastic, or that it had been strained beyond the elastic limit.

In springs, as commonly used, the displacement is primarily along one line so that one coordinate suffices. If $x$ is the displacement coordinate measured from the state where there is no force applied to the spring, or the spring applies no force, the force associated with the extension or contraction of the spring is clearly represented by a function $f(x)$ of $x$. Although in ordinary springs this is proportional to $x$, nonlinear springs are now used in industry where, for example,

$$f(x) = kx^2$$

for a constant $k$. In any case the slope of the load-deflection curve is called the *spring scale*. In what follows the springs are linear except where a remark is made to the contrary. See Figure 5 illustrating types of springs in common use.

A plastic material will generally not come back to its original shape when the initial forces are reapplied after some other distribution of forces. The forces associated with such a body thus depend on the past history of the body.

We can write (1.5) as

$$F + F_m = 0 \tag{1.45}$$

for the *inertial force* $F_m$ equal to $-ma$. Following d'Alembert we thus think of (1.5) as a relation which says that a sum of forces vanishes, one of the forces being the applied force. We say that the particle is in *dynamic equilibrium*, that is, the sum of the forces associated with the particle vanishes. For a particle traveling along an $x$-axis $F_m$ is, of course, represented by $-mx''$.

In the discussion above the forces that act on particles due to other particles are position and velocity forces. There is also a force which depends on acceleration and is not an inertial force. Due to electromagnetic induction we can associate a force on an electric charge $q_1$ due to the acceleration $a_2$ of an electric charge $q_2$, where this force is proportional to the product of $q_1$, $q_2$, and $a_2$, and inversely proportional to the distance $r$ between the charges. We shall not need to make explicit use of this force in the developments to follow. Since by the calculus we reduce the study of continuous distributions of matter to that of particles, we can extend the treatment of particles to bodies spread out over space, and thus obtain the forces which can arise in the study of bodies. We must know what kinds of force to look for in a physical problem in order to be able to set it up in mathematical form.

We are including electric charges and magnetic poles in this discussion when we refer to particles.

*The force on a particle is proportional to the mass or charge of this particle.*

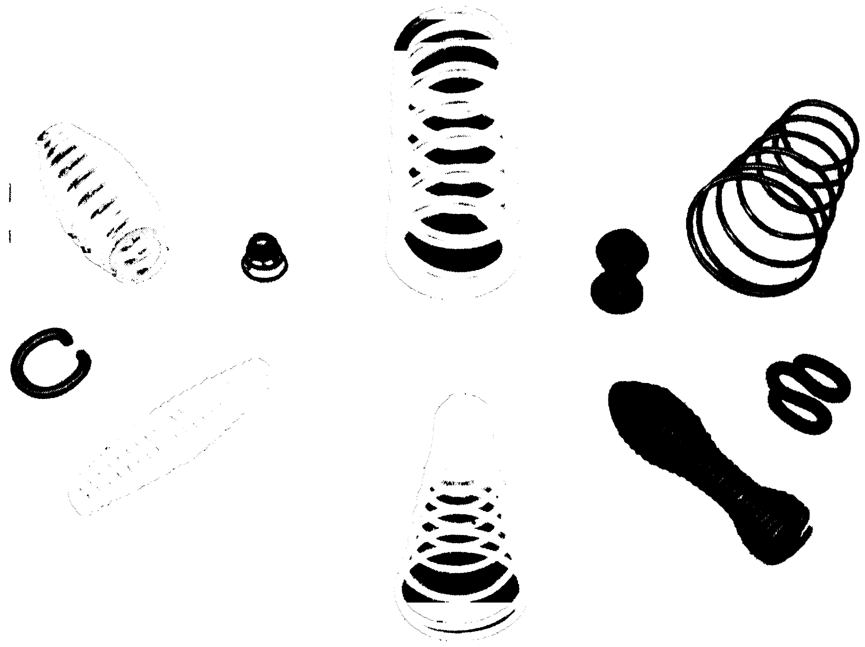

**Fig. 5. An assortment of industrial linear and non-linear springs. Courtesy of the William D. Gibson Company.**

Suppose that whenever the elements of a given body are replaced by particles, only position forces act on these particles individually. We would then expect the resultant of the forces on the body to be a function of the coordinates of the reference points in the body. That this is true follows from the type of argument below in the special case of a plate.

We consider a rigid plate $B$ in motion, as in Figure 6, relative to an $xy$-Cartesian coordinate system. The point $\bar{O}(\bar{x}, \bar{y})$ is a fixed point in the plate, and $\bar{O}A$ is a polar axis rigidly attached to the plate. We let $\omega$ designate an angle between $\bar{O}A$ and the positive direction of the $x$-axis. The position of each point in the plate is known as soon as $\bar{x}$, $\bar{y}$, and $\omega$ are given. Each point $P(x, y)$ on the plate has polar coordinates $(r, \theta)$ that are fixed for the point, and do not change with the motion of the plate.

We have the following relations

$$\begin{aligned} x &= \bar{x} + r\cos(\theta + \omega), \\ y &= \bar{y} + r\sin(\theta + \omega). \end{aligned} \tag{1.46}$$

We shall suppose, for simplicity, that if a particle is placed at a point in the plane, this particle is under the influence of a position force due to the position of this particle relative to $O$. Thus we might have a mass particle at $O$, or electric or magnetic charge, or a combination of these things. Actually, magnetic charges occur in pairs so that in the case of magnetism we would have to consider dipoles rather than magnetic charges (see Chapter III). We have a force $F\rho\Delta A$ on the element of the plate with mass or charge $\rho\Delta A$, where $\Delta A$ is the area and $\rho$ the density of the element. The quantity $\rho$ might, for example, be electric charge density. If we are treating gravitational effects, $\rho$ is mass density. If we concentrate the mass or charge $\rho\Delta A$ at a single point $P(x, y)$ on the plate, the force acting at $P$ is $F(x, y)\rho\Delta A$ for a vector function $F(x, y)$ of $x$ and $y$ (actually of $x^2 + y^2$). The density $\rho$ is a function of $r$ and $\theta$. The component of $F$ in the $x$-direction is represented by a real function $F_x(x, y)$. We take $dA = rdrd\theta$. By the calculus the resultant $R_x$ of all of the forces acting on the plate $B$ in the $x$-direction is given by

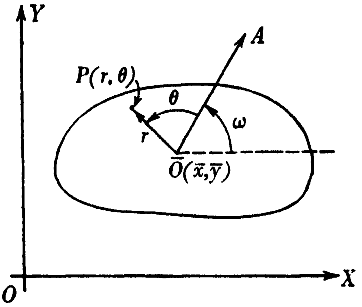

Fig. 6. Plate *B* in motion.

$$R_x = \iint_B \rho F_x(x, y) r dr d\theta. \tag{1.47}$$

By (1.46) relation (1.47) becomes

$$R_x = \iint_B \rho F_x[\bar{x} + r\cos(\theta + \omega), \bar{y} + r\sin(\theta + \omega)] r dr d\theta. \tag{1.48}$$

The integral in (1.48) is clearly a function $R_x(\bar{x}, \bar{y}, \omega)$ of $\bar{x}$, $\bar{y}$, and $\omega$. Similarly for the resultant $R_y$ in the $y$-direction. If $\bar{O}$ is the center of mass of the plate, the motion of this point is thus determined by position forces dependent on the coordinates $\bar{x}$, $\bar{y}$, and $\omega$ of position only, and not dependent in any other way on the coordinates of position of the points on the plate. Of course, the forces depend on the nature of the source at the origin.

If now the plate $B$ is under the influence of a set of particles or continuous distribution of mass (or electric, or magnetic charge), summation over the set of particles or integration of the continuous mass source of force, where the source coordinates are known, still yields a resultant force which is a function of $\bar{x}$, $\bar{y}$, and $\omega$, and not dependent in any other way on the coordinates of position of the points of the plate. The summation or integration in this case is over coordinates of points of the source and does not affect $\bar{x}$, $\bar{y}$, or $\omega$.

Equation (1.5) applied to the $x$ and $y$-directions yields two equations of motion for the center of mass $\bar{O}$, which are sufficient to determine the motion of this point, with, of course, suitable initial conditions to be explained in a later section. We need another equation to determine $\omega$. As we shall show later, an analogue of (1.5) for rotatory motion involving torque and moment of inertia yields this equation. Analogy between linear and angular motion shows that in our present problem the torque is a function of position only.

The reader can easily prove that in general, *if a rigid body is acted on by position forces, these forces are functions only of the position coordinates* used to describe the state of the body, and do not involve these coordinates in any other way.

### 1.12. Position-velocity forces on rigid bodies.

We have shown already that if a rigid body is moving through a medium other than empty space, the collisions of the body with the molecules of the medium yield a damping force that is a function of the velocity of some reference point in the body. As also remarked above, the body may be acted on by forces due to moving electric charges and dependent on velocity (as well as position). This would be true if we had to consider the electric and magnetic properties of the body. We would expect that the resultant force on the affected body due to moving electric charges depends on the reference coordinates and their derivatives only (the body and charges being given), inasmuch as the derivative of a function of these coordinates will be a function of these coordinates and their derivatives. That this is actually the case is a consequence of the type of argument below applied to the plate $B$ of Figure 6 subject to the forces of moving charges.

We suppose that a single electric charge is placed at $O$, such as an electron, and that the plate is moving in the plane relative to $O$. This is the same as having the charge move relative to the plate. Differentiating the relations (1.46) we have

$$\begin{aligned} x' &= \bar{x}' - r\omega' \sin(\theta + \omega), \\ y' &= \bar{y}' + r\omega' \cos(\theta + \omega), \end{aligned} \tag{1.49}$$

where the primes denote derivatives with respect to time.

To simplify the argument we shall suppose that the plate is magnetized with *magnetic density* $\rho$. This is of mathematical interest only because, in order to agree with nature, we should treat dipoles rather than magnetic charges (see Chapter III). If we concentrate an element $\rho\Delta A$ of the plate at a point $P(x, y)$ on the plate, the force on the element, as we shall see in Part II, is actually perpendicular to the plane of the plate, and is a function of the velocity of the point $P$, as well as the distance from $O$ to $P$ (this force is also proportional to $\rho\Delta A$). This means that the components of the force on the element are represented by functions $\rho F_x(x, y, x', y')\Delta A$ and $\rho F_y(x, y, x', y')\Delta A$ as far as the coordinates of $P$ are concerned. It follows that the resultant of the $x$-components of the forces acting on the plate, due to motion relative to the charge at $O$, is represented by $R_x$ where

$$R_x = \iint_B \rho F_x(x, y, x', y')r\,dr\,d\theta. \tag{1.50}$$

A similar formula holds for $R_y$. In view of (1.46) and (1.49) the force $F$ acting on the plate is a function of $\bar{x}$, $\bar{y}$, $\omega$, $\bar{x}'$, $\bar{y}'$, and $\omega'$.

This argument extends to more complicated sources of force due to moving charges. In any case the resultant force is a function of coordinates of position and their derivatives, the coordinates being any set used to determine the plate. Such a force is a *position-velocity force* dependent on both position and velocity. Similar results hold for the effect of moving charges (like currents in a wire) on other moving charges or magnets in the general case.

In Part II we shall treat *fields of force*. We have been working with such fields in this and the preceding section.

**1.13. Dry friction.** When one solid moves over another we have *dry friction* between the solids. This type of friction is important in the analysis of unoiled parts of machinery which rub each other. A typical curve is shown in Figure 7 for a Cartesian distance coordinate $x$ versus time $t$. The friction force in the case of linear motion is numerically large when the motion begins, and drops off rapidly as the speed of one solid relative to the other increases, being fairly constant for high speeds. When there is no motion this force can be anything in a certain finite range, not shown. For the average engineering application reasonable results can be obtained by replacing the function $F$ representing the friction force, whose magnitude is given in Figure 7, by

$$\begin{aligned} -K \leqq F \leqq K &\text{ for } x' = 0, \\ F = -K \quad &\text{ `` } x' > 0, \\ F = K \quad &\text{ `` } x' < 0. \end{aligned} \tag{1.51}$$

The force represented by $F$ of (1.51) is pictured in Figure 8. The friction force always opposes the motion. Thus when $x' > 0$, this force is directed toward the negative half of the $x$-axis and is represented by a negative function. We have here a case of a velocity force not uniquely determined when $x' = 0$.

These results extend readily to nonlinear motions.

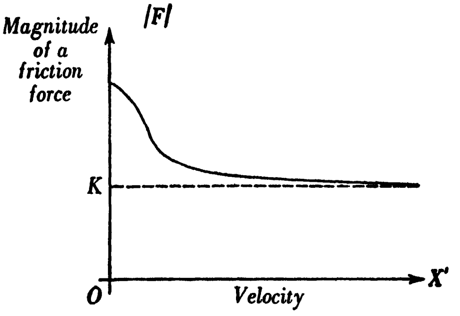

**Fig. 7. Dry friction $F$.**

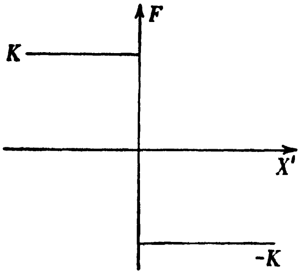

**Fig. 8. Dry friction.**

**1.14. Forces on rigid bodies.** We have been considering physical forces only, in contrast to forces applied by muscular effort. We shall also exclude forces which can be ascribed to chemical reactions, that is, the change of chemical energy to energy of motion, and certain other energy changes such as those involving heat. Such forces can vary in rather general ways depending on the nature of the reaction. *We restrict ourselves to the forces due to mechanics and electromagnetism.* In Principle 1.1 to follow *we exclude the acceleration force of Section 1.11* on an electric charge $q_1$ due to the acceleration $\boldsymbol{a}_2$ of an electric charge $q_2$ *because in engineering we are generally concerned with currents rather than individual charges.* Our analysis shows the following principle to be valid.

***Principle* 1.1.** *Exclude from consideration the acceleration force of Section 1.11 on an electric charge. The physical forces which act on rigid bodies, where these forces are independent of past history, and are due to mechanics and electromagnetism only, are functions of the coordinates of position of these bodies and the first derivatives of these coordinates with respect to time, and do not otherwise involve these coordinates.*

In the statement of this principle the functions are understood to be vector functions. If we work with components in the directions of the axes of a Cartesian coordinate system, these functions are real functions.

By forces "due to mechanics," that is, *mechanical forces,* we mean forces existing because of the masses in the problem. Electromagnetic forces are due to electric charges and magnetic poles.

*Principle 1.1 covers the forces an engineer may expect to find in most physical problems.*

We have arrived at Principle 1.1 by an analysis of the forces due to the presence of gravitational, electric, and magnetic fields, and due to collision of these bodies with particles of surrounding media (also, such molecular forces as those of Van de Waals are position forces). For the sake of brevity we have omitted some proofs which follow the lines already indicated.

We have stated Principle 1.1 to include electric and magnetic effects because these often occur in conjunction with mechanical aspects of a problem. Because of the principle of superposition of forces we may treat the mechanical and electromagnetic effects separately, and combine these in setting up the differential equations of motion. Thus, if $F_1$ is the resultant of the mechanical forces acting on a rigid body, and $F_2$ the resultant of the electromagnetic forces, and if $M$ denotes the mass of the body and $a$ the acceleration of its center of gravity, we have the equation of motion

$$F_1 + F_2 = Ma.$$

Principle 1.1 includes forces that are constants in the problem, such as, perhaps, the force of gravity.

Principle 1.1 simplifies the problem of finding the forces acting on a physical system. It is clearly unnecessary to look for forces that are functions of third derivatives of coordinates of position; or as another example, it is unnecessary to look for forces proportional to squares of the second derivatives of position.

*If a rigid body is being translated in the x-direction only, by Principle 1.1 we may expect the function representing the resultant force to be of one of the forms*

$$F(x), \quad F(x'), \quad F(x, x'),$$

*and if forces due to currents (moving charges) are excluded we have forces of the first two types only.* This is particularly important in the solution of engineering problems. For bodies rotating about axes, $F(x)$ and $F(x')$ become the torques $T(\theta)$ and $T(\theta')$ for the angle of position $\theta$, and the derivative $\theta'$ of $\theta$.

Except for the acceleration force of Section 1.11 on an electric charge the second time derivatives of position coordinates occur only in inertial forces, which are not thought of as applied forces.

It will appear from Part II that a force on a magnetic material may depend on the past history of the material.

**1.15. Units.** The magnitude of the force of attraction of the earth on a body is called its *weight*. It follows that if $w$ is the weight of a body and $m$ its mass, we can obtain $m$ from the relation

$$m = \frac{w}{g} \tag{1.52}$$

$g$ being the magnitude of the acceleration of gravity. We shall employ the following units weight in *pounds*, distance in *feet* time in *seconds*, whence mass is in *slugs*.

Engineers usually avoid the term *slug*, preferring the dimension $F$ of force to that of mass, as remarked in Section 1.5.

**1.16. Dashpots.** Many mechanical systems can be considered to be made up of masses, springs, and dashpots, and are often constructed directly from such elements. A *dashpot* is composed of a piston which moves in a cylinder filled with a nonrigid substance that is permitted to slide past the piston. This substance may be air as in an electric bread toaster, or oil as in the shock absorbers of an automobile. If the piston is not moved, the medium in which it is immersed exerts no force on the piston. The faster the piston is moved, the greater is the force. For ordinary dashpots this force is given by

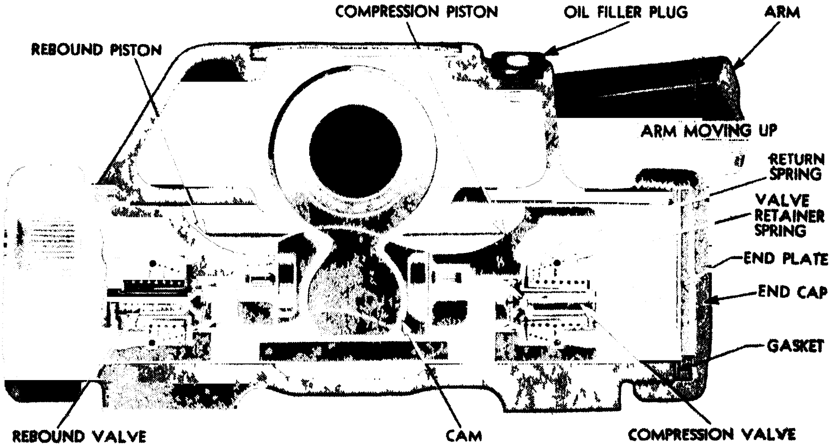

**Fig. 9. Automobile shock absorber. Courtesy of the General Motors Corporation.**

$$F = -cx' \tag{1.53}$$

for a positive number $c$ and velocity of the piston designated by $x'$.

If there is a damping force, represented by $F$ of (1.53), acting on a body, we may think of this as caused by a dashpot. Similarly, a position force may be thought of as due to a spring.

An industrial example of a dashpot is given in Figure 9.

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2. J. L. Synge and B. A. Griffith, *Principles of Mechanics*, McGraw-Hill, New York, 1942.
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# CHAPTER

# 2

# *Applications and further development of the theory of the mechanics of rigid bodies*

**2.1. Example of a spring-mass system.** We are ready to set up mechanical systems in mathematical form. We shall begin with a simple example. We have Figure 10 where a linear spring with scale $k$ is attached to a support. The other end is connected to a body $B$ with mass $m$ whose position is determined by a coordinate $x$. An arrow is used to indicate the positive direction for $x$. Let us suppose that the spring-mass system hangs vertically from the support. We shall take the origin of the $x$-axis at the position occupied by the body $B$ when $B$ is at *rest*. By definition this occurs when $x'$ is identically zero, that is,

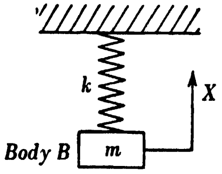

**Fig. 10. A simple mechanical system.**

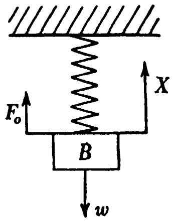

**Fig. 11. A spring-mass system.**

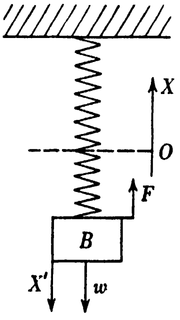

**Fig. 12. Body $B$ below its equilibrium position.**

$$x' \overset{t}{\equiv} 0. \tag{2.1}$$

The situation is pictured in Figure 11 where the weight $w$ of the body $B$ is balanced by the force, represented by $F_0$, due to the elongation of the spring. For the position of rest of $B$, equation (1.5) reduces to

$$F_0 - w = 0, \tag{2.2}$$

the force acting on $B$ being represented by $(F_0 - w)$, and the acceleration being zero.

We shall consider any state of the body $B$. By *state* we mean the physical situation in which the body finds itself at a particular instant. As an example, we take Figure 12 with the body $B$ below the origin, and traveling downward. In this figure $F$ represents the *spring* (position) *force.*

We recall that if $x'$ is positive the *velocity vector* is pointed toward the positive end of the $x$-axis. It follows from vector differentiation that if $x''$ is positive the acceleration vector is directed upward also. Hence the force represented by $mx''$ is *directed upward if* $x''$ *is positive*, and is called *positive.*

It is convenient to point the velocity vector upward when $x'$ is positive, since the direction of motion is upward in this case. We could, of course, violate custom, and point the vector the other way, obtaining logical mathematics as a result. In a mathematical study we have a right to take the positive sense any way we please.

Since $(F - F_0)$ is positive (the spring is stretched downward from equilibrium) and $kx$ is negative for the state of Figure 12, $k$ being understood to be positive, it follows that

$$F = F_0 - kx. \tag{2.3}$$

By (1.5) we have

$$F_0 - w - kx = mx''. \tag{2.4}$$

In writing this equation the body $B$ is treated as if it were a *free* body. From (2.2) and (2.4) we obtain

$$mx'' + kx = 0. \tag{2.5}$$

If we had started with any other state, we would have gotten the same second order differential equation (2.5), that is, analysis of the ± signs would have led to the same signs as used in (2.5).

The equation (2.5) is not enough to determine the motion of the body $B$. We need initial conditions to complete the mathematical picture. It is convenient to know what $x$ and $x'$ are for one state, for example, when $t = 0$. Let $x_0$ and $x'_0$ be the values of $x$ and $x'$ respectively, when $t = 0$. Then the *initial conditions*

$$t = 0, \;\; x = x_0, \;\; x = x'_0 \tag{2.6}$$

with (2.5) form a mathematical system from which the position of $B$ for any instant can be determined.

In place of (2.6) we may use other sets of conditions such as

$$t = 0, \;\; x = x_0; \;\; t = t_0, \;\; x = x_1, \;\; t_0 \neq 0. \tag{2.7}$$

The advantage of (2.6) over (2.7) is that (2.6) depends on one state only, whereas (2.7) is determined from two states.

By the theory of differential equations the general solution of (2.5) involves two arbitrary constants whose values can be found from (2.6), (2.7) or other information about particular states.

Experience shows that the coordinates of position vary continuously with time, that is, *a body cannot instantaneously jump from one position to another.* However, we may get very satisfactory results in setting up initial conditions if, when an extremely rapid motion occurs in a problem in conjunction with a slow motion, we assume that the rapid motion takes place instantaneously. Thus if a part of a machine changes position in 0.01 second, and we are concerned with the motion of the parts of the machine over minutes or even over one second, we would probably be justified in supposing that the change of position of the given part occurred instantaneously.

It is also sometimes convenient to assume that velocities change instantaneously, as when an automobile traveling at 50 *miles per hour* (mph), crashes into a stone cliff, and we are concerned with the motion of the occupants. In this case we might suppose that the speed of the automobile changes instantly from 50 miles per hour to zero miles per hour.

**2.2. Force deviations from equilibrium.** The force of gravity does not even occur in (2.5) and (2.6). It could have been ignored in transforming the physical system into mathematical form had we measured our forces as deviations from equilibrium, equilibrium in this case being the state where the body B is at rest ($x' \equiv 0$). This is a case of *static equilibrium,* that is, a state where the derivatives of the coordinates of position with respect to time vanish identically. We have illustrated the principle to follow.

***Principle* 2.1.** *In setting up the differential equations of motion for a body under the influence of a system of forces, these forces may be measured as deviations from the forces which exist when the body is in static equilibrium. In particular, constant forces (having the same value as in static equilibrium) can be ignored.*

Principle 2.1 often gives a great simplification of the mathematics of a physical problem. We shall show why this principle is valid.

Let us consider a particle of mass $m$ and acceleration $\boldsymbol{a}$. Its equation of motion is

$$\boldsymbol{F}_0 + \boldsymbol{F} = m\boldsymbol{a}, \tag{2.8}$$

where $\boldsymbol{F}_0$ is the total force acting on the particle when it is at rest, and $(\boldsymbol{F}_0 + \boldsymbol{F})$ is the force acting on the particle at the instant being considered. However, in static equilibrium $\boldsymbol{a} = 0$, whence $\boldsymbol{F}_0 = 0$. This argument extends to any physical system.

Principle 2.1 can be extended to a slightly *more general form.* We can measure our forces and accelerations as deviations from what they are in any one state in which no point in the body is accelerating.

Under the influence of the forces, for which we may wish to treat a particular problem, there may be no equilibrium state to which Principle 2.1 is applicable. For engineering purposes this principle is generally satisfactory as stated.

In the typical engineering problem gravity is constant, and can therefore be ignored as far as the study of motion is concerned. Since static equilibrium problems are relatively easy compared to problems concerning motion, we shall concentrate on the latter.

If in Figures 10–12 the spring is changed to a horizontal position, or placed at a slant, we may still ignore the force of gravity in view of Principle 2.1. Of course, that the force of gravity may be ignored when the spring is horizontal also follows from the fact that this force acts perpendicularly to the direction of motion of the body $B$, and therefore has no effect on the motion of $B$. We assume here that any sag in the spring is negligible.

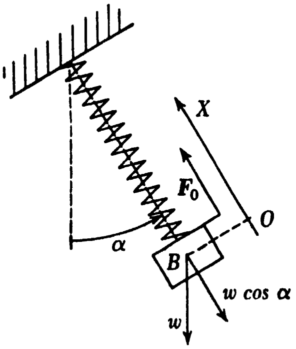

**Fig. 13. Spring at an angle with the vertical.**

No matter at what angle the spring is placed relative to the vertical, there is a state for which $x' \equiv 0$, in which case the component of gravity along the spring is exactly balanced by the force in the spring due to stretching. Such a situation is pictured in Figure 13, where the direction of motion makes an angle $\alpha$ with the vertical, and the component of the force of gravity along the line of motion has the magnitude $w \cos \alpha$, balanced by an equal and opposite spring force $F_0$.

If the body $B$ in Figure 13 is displaced by a directed distance $x$ from equilibrium, the spring force changes by an amount represented by $F$, where $F = -kx$. Applying (1.5) we immediately obtain (2.5). This is the procedure commonly used in *industrial practice.*

The treatment of the spring-mass system above is premised on the assumption that the spring is without mass, and is linear over the range of motion considered. If the mass of the spring must be taken into account, we usually include one-half the mass of the spring with the mass of the body.

We shall illustrate the preceding development with typical numerical cases.

**2.3. Clocks.** Several clocks have been built with a timing mechanism in the form of the mechanical system of Figure 10. Let us suppose that the bob weighs 0.1 pound (lb), the weight of the spring is 0.02 pound, the spring scale $k$ is 0.12 pound per inch (lb per in), and the length of the unloaded spring is 4 inches.

Measuring the position $x$ from equilibrium as in Figure 11 we have the spring force $F$ in pounds given by

$$F = -0.12x$$

for $x$ in inches. Following standard engineering practice we take one-half the mass of the spring and add it to that of the bob. The bob then weighs 0.11 pound. The mass $m$ of the bob is then 0.11/386 pound second squared per inch (lb sec$^2$ per in), which is 0.000285 pound second squared per inch. Now $mx''$ is 0.000285 $x''$. Equating $F$ to $mx''$ we have

$$0.000285x'' = -0.12x, \tag{2.9}$$

which is equivalent to equation (2.5).

Although it is not the purpose of the author to teach the solution of mathematical problems in this book, we shall treat (2.9) for a few sets of initial conditions to illustrate typical solutions of physical problems.

We let $e$ be the Napierian base of logarithms. Substituting $x = e^{\alpha t}$ in (2.9) and dividing both sides of the resulting equation by $e^{\alpha t}$, we obtain the auxiliary equation

$$0.000285\alpha^2 + 0.12 = 0, \tag{2.10}$$

whose solutions for $\alpha$ are approximately $\pm 20j$, where $j$ is $\sqrt{-1}$. We use $j$ instead of $i$ to avoid confusion with the symbol for electric current to be used later. The general solution of (2.9) is now

$$x = C_1e^{-20jt} + C_2e^{20jt} \tag{2.11}$$

for arbitrary numbers $C_1$ and $C_2$ to be determined from initial conditions.

It is convenient to use the mathematical law

$$e^{j\phi} = \cos \phi + j \sin \phi \tag{2.12}$$

valid for a real number $\phi$. By (2.12) relation (2.11) becomes

$$x = (C_1 + C_2) \cos 20t + j(C_2 - C_1) \sin 20t. \tag{2.13}$$

Since for any numbers $A$ and $B$ we can solve the equations

$$\begin{aligned} C_1 + C_2 &= A, \\ j(C_2 - C_1) &= B \end{aligned} \tag{2.14}$$

for $C_1$ and $C_2$, and conversely get $A$ and $B$ for any values given to $C_1$ and $C_2$, we may write (2.13) as

$$x = A \cos 20t + B \sin 20t \tag{2.15}$$

whence

$$x' = 20(B \cos 20t - A \sin 20t). \tag{2.16}$$

Let us take the initial conditions

$$t = 0, \quad x = 0, \quad x' = 0. \tag{2.17}$$

Substituting the values of (2.17) in (2.15) and (2.16) we have $A = B = 0$, so that $x \equiv 0$. The bob starts at rest and will remain at rest.

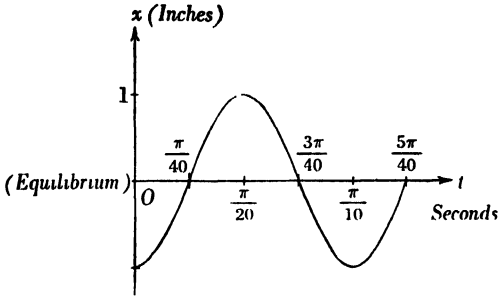

Fig. 14. Motion of a bob.

Let us suppose that the bob is pulled down one inch from equilibrium, and then released. The initial conditions in this case are

$$t = 0, \quad x = -1, \quad x' = 0, \tag{2.18}$$

the unit of distance being 1 inch. Putting these values in (2.15) and (2.16) we find that $A = -1$, $B = 0$, so that the motion of the bob is described by

$$x = - \cos 20t. \tag{2.19}$$

The graph of (2.19) is given in Figure 14. The bob will oscillate between 1 inch above and 1 inch below the equilibrium position with a period of $\pi/10$ seconds, and frequency of $10/\pi$ cycles per second. We have here a case of *simple harmonic motion.*

The equilibrium position is obtained by equating the weight 0.11 pound to the force $0.12y$ pounds due to the $y$ inches the spring is stretched from equilibrium. Solving we find that $y = 0.92$ inch, so that the length of the spring in the equilibrium (static) position, for which $x = 0$, is 4.92 inches. The oscillations occur about this position.

We have taken the mass of the spring unusually high to show how it enters into the problem. In actual practice it would be very much smaller.

It is important for the reader to familiarize himself with the methods of solution of the equations of motion of physical systems so that he will know what to look for in stating his problem in mathematical form. He will normally want enough information on the mathematical formulation to determine the solution uniquely. Mathematical theories of the existence of solutions shed considerable light on this matter.

**2.4. Automobile without shock absorbers.** As a second numerical example let us consider an *automobile with shock absorbers removed.* The body is connected to the axles by springs which need not be helical, but may be leaf springs, as far as the argument to follow is concerned. We shall suppose that the automobile is bouncing straight up and down, and that the movement of the body relative to the axles is large compared to that of the axles relative to the ground, that is, the give of the tires is negligible. We may replace all the body springs by one spring whose scale is the sum of the scales of the given springs, as shown in Figure 15.

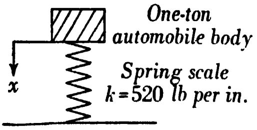

**Fig. 15. Automobile bouncing on a road.**

We shall suppose that the weight of the automobile body is 1 ton or 2,000 pounds. The spring scale is taken to be 520 pounds per inch. The body at the start will be taken 1 inch above equilibrium, and traveling downward at a speed of 1 foot per second (ft per sec). We measure our variables from equilibrium where $x = 0$, and let the positive sense for $x$ be downward.

That the mass of the springs can be neglected is obvious. We shall carry through the computation with inches.

Since the mass has the dimension $FT^2/L$, the mass of the automobile is 2,000/386 pound seconds squared per inch, that is, about 5.2 pound seconds squared per inch. The spring force is $-520x$ pounds for $x$ inches, so that

$$5.2x'' + 520x = 0. \tag{2.20}$$

The initial conditions are

$$t = 0, \quad x = -1, \quad x' = 12. \tag{2.21}$$

The solution of (2.20) is of the form

$$x = A \cos 10t + B \sin 10t. \tag{2.22}$$

By $t = 0$, $x = -1$, we have $A = -1$, whereas $t = 0$, $x' = 12$ yields $B = 1.2$.

The automobile body will thus bounce up and down with a frequency of $10/2\pi$ or roughly 2 cycles per second. The amplitude of the bounce will be $\sqrt{A^2 + B^2}$ inches, in this case about 1.6 inches.

The examples of this section and the preceding one illustrate how *from one state and ordinary differential equations of motion we can predict the other states of a physical system.*

## 2.5. Spring-dashpot-mass system.

In the examples of the two preceding sections the motion will continue forever. In actual practice damping would eventually bring the systems to a halt, so that over a long period of time damping forces cannot be neglected. A simple example in damping follows.

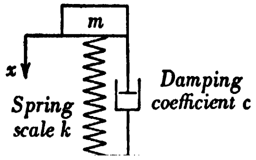

**Fig. 16. Spring-dashpot-mass system.**

We consider the system of Figure 16 where a body with mass $m$ is acted on by spring and dashpot forces represented by $-kx$ and $-cx'$. We have the differential equation of motion

$$mx'' + cx' + kx = 0 \tag{2.23}$$

coming from (1.5).

As a numerical example we shall treat an automobile with *shock absorbers*. We shall take the same physical situation as in Figure 15 except that shock absorbers (dashpots) are added, for which the combined damping coefficient $c$ is 83.2 pound seconds per inch (lb sec per in). In place of (2.20) we have

$$5.2x'' + 83.2x' + 520x = 0, \tag{2.24}$$

for which the auxiliary equation is

$$5.2\alpha^2 + 83.2\alpha + 520 = 0. \tag{2.25}$$

Solution of (2.25) yields approximately

$$-8 \pm 6j.$$

The general solution of (2.24) is thus

$$x = e^{-8t}(A \cos 6t + B \sin 6t). \tag{2.26}$$

The presence of $e^{-8t}$ means that the shock due to the initial conditions (2.21) is quickly damped out.

The solution of the system is

$$x = e^{-8t}(-\cos 6t + \tfrac{2}{3}\sin 6t). \tag{2.27}$$

The relation (2.27) is graphed in Figure 17. The motion is seen to be oscillatory but well damped.

The analysis of the above example can also be applied to an *airplane* whose landing gear is equipped with springs and shock absorbers. In fact, the airplane might in a practical case make contact with the ground at the instant when its downward speed is 1 foot per second. The wheels stop their downward motion instantly, for rough practical purposes, and may be considered to be fixed as far as vertical motion is concerned. Although the wheels are fixed at contact the plane is not, but rather has the velocity of 1 foot per second downward. If at the instant of contact the plane is 1 inch above its final equilibrium we have exactly the initial conditions of (2.21). If the elements of the plane are properly chosen we may also have (2.24) valid.

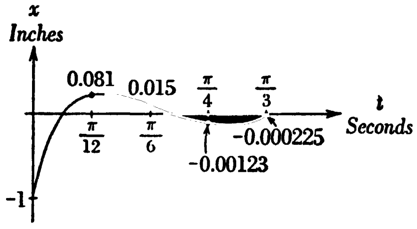

**Fig. 17. Curve for the motion of an automobile with shock absorbers.**

## 2.6. Instrument on a rubber foundation.

Many scientific instruments are placed on blocks of rubber to absorb the shocks caused by the passing of nearby trains or other traffic. Experiments show that a rubber block behaves like a spring and dashpot as shown in Figure 18. In this example the rubber block, represented by the spring and dashpot, rests on a base that vibrates. The number $d$ designates the distance in inches from the base to the point $O$ of equilibrium for the top end of the spring, that is, the distance from the base to the instrument when the system is at rest. The variable $y$ is the distance in inches from a fixed reference plane to the point $O$. The point $O$ is understood to move with the base so that $O$ is always $d$ inches above the base.

By a *fixed* reference plane we mean that the motion of the system is studied relative to this plane. An $x$-axis is introduced so that the distance from the reference plane to the instrument is $(x + y)$ inches. To say that the base vibrates is the same as saying that the point of equilibrium oscillates; in fact since $d$ is constant they both oscillate in the same way.

The spring force in pounds acting on the instrument, measured from that existing at equilibrium, is taken to be $-100x$. The damping force in pounds is $-10x'$. In Figure 18 we have pictured a state where the instrument is above the equilibrium position. From this state it is readily seen that the signs in the formulas for the spring and damping forces are taken correctly. The acceleration of the instrument being $(x + y)''$ and its weight being 64 lb (mass $\frac{64}{32}$ slugs) the equation of motion of the instrument (as a free body) is

$$\tfrac{1}{6}(x + y)'' + 10x' + 100x = 0. \qquad (2.28)$$

The coefficient $\frac{1}{6}$ comes from dividing slugs by 12, in converting from feet to inches.

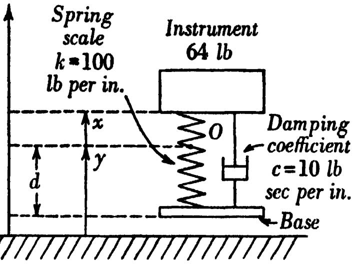

Fig. 18. Instrument resting on a rubber block.

With $u = x + y$ we can write (2.28) in the preferable form

$$u'' + 60u' + 600u = 60y' + 600y. \qquad (2.29)$$

Let us suppose that we have a typical sinusoidal oscillation of the form

$$y = K + 0.3 \sin 100\pi t \qquad (2.30)$$

for a real number $K$. For a properly chosen reference plane the number $K$ vanishes. We shall, however, carry through the analysis for an arbitrary $K$. The oscillatory part of $y$ in (2.30) has an amplitude of 0.3 inches and frequency of 50 cycles per second. Substituting $y$ from (2.30) in (2.29) we derive

$$u'' + 60u' + 600u = 600K + 180(10\pi \cos 100\pi t + \sin 100\pi t), \qquad (2.31)$$

which is a nonhomogeneous linear differential equation of the second order.

From the theory of differential equations the general solution of (2.31) is of the form

$$u = Ae^{(-30+10\sqrt{3})t} + Be^{-(30+10\sqrt{3})t} + f(t), \qquad (2.32)$$

where $f(t)$ is a particular solution of (2.31), and $A$ and $B$ are numbers that can be determined from initial conditions. Since, for practical values of $A$ and $B$, the transient terms die rapidly, we do not need initial conditions unless we are interested in what happens at the start of the motion. Following the customary procedure of handling *vibrations* we write

$$u = f(t) \tag{2.33}$$

keeping the *steady state term* $f(t)$ only. The function $f(t)$ is determined by the forcing function. From the theory of differential equations $f(t)$ is of the form $K + a \cos 100\pi t + b \sin 100\pi t$ for numbers $a$ and $b$, which can be obtained by substituting this formula for $u$ in (2.31) and equating corresponding coefficients. Solution yields $a = -0.06$, $b = 0.009$. The motion of the instrument is thus sinusoidal with a frequency of 50 cycles per second. The amplitude $\sqrt{a^2 + b^2}$ turns out to be about $\frac{1}{17}$ inch. The rubber block actually reduces the amplitude of vibration of the instrument to approximately one-fifth the value it would have if the instrument were rigidly connected to the oscillating base.

The frequency of the response of a linear system (described by linear differential equations) is the same as the forcing frequency. This is not always true for nonlinear systems.

**2.7. Automobile with shock absorbers and elastic tires (motion with two degrees of freedom).** In the examples above, each physical system was determined by the value of one variable. We were thus treating systems with *one degree* of freedom, so that *one differential equation* plus initial conditions sufficed to describe the motion. When $n$ variables are needed we have a system with $n$ degrees of freedom. An example with two degrees of freedom follows.

In the automobile illustrations above we neglected the elastic properties of the tires. The situation we wish to consider now is that of Figure 19, where the top spring and dashpot have the same meaning as in Section 2.5 with the automobile body of mass $m_1$ resting on them. The axles, wheels, and transmission are lumped together and have mass $m_2$. The bottom spring and dashpot represent tires.

The system shown in Figure 19 is taken in static equilibrium to define the distances $d_1$ and $d_2$. The origins of the $x$ and $y$ coordinate systems are always distances $d_1$ and $d_2$ from the $m_2$ mass and ground, respectively.

By applying (1.5) to each mass individually we obtain the equations of motion of the system, namely,

$$\begin{aligned} m_1(x + y)'' + c_1x' + k_1x &= 0, \\ m_2y'' + c_2y' + k_2y - c_1x' - k_1x &= 0. \end{aligned} \tag{2.34}$$

The two equations correspond to the two degrees of freedom. Here $x$ and $y$ are coordinates of the $m_1$ and $m_2$ masses respectively.

To determine the position of each point in the system it is necessary to adjoin to (2.34) a set of initial conditions, such as

$$t = 0,\ x = x_0,\ x' = x_0',\ y = y_0,\ y' = y_0' \qquad (2.35)$$

giving the position and velocity of the bodies of Figure 19 at $t = 0$.

Equations (2.34) are an immediate consequence of (1.5) with $m = m_1$ and $m = m_2$, respectively. We can obtain an equation of motion by considering the physical system composed of the two masses and the spring and dashpot joining them, letting $F$ in (1.5) be the external force applied to this system, and using the sum of products of mass by acceleration for the two bodies in place of $ma$.

The bottom spring and dashpot of Figure 19 exert on the system an external force represented by $-(c_2y' + k_2y)$. The inertial force due to $m_1$ is represented by $-m_1(x + y)''$, and that due to $m_2$ is represented by $-m_2y''$, whence adding the external force to these inertial forces we obtain zero. Thus

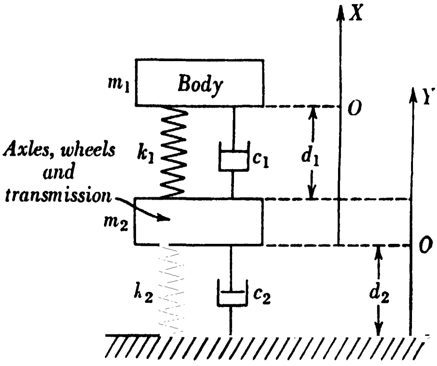

**Fig. 19.** Automobile with shock absorbers and elastic tires.

$$m_1(x + y)'' + m_2y'' + c_2y' + k_2y = 0. \qquad (2.36)$$

Equation (2.36) and the first equation of (2.34) form a convenient pair of equations of motion for the system.

### 2.8. Sliding weight subject to dry friction.

In the examples above the forces have been expressed by continuous functions. We shall show how a simple physical system with dry friction can be treated.

Let us consider a body $B$ which slides in an unoiled groove and is supported from below by a nonlinear spring, as shown in Figure 20. Let $m$ be the mass of $B$, and $F_c$ a function representing the friction force on $B$. Let us suppose that $F_c$ is equal to $F$ of Figure 8 when $B$ is in motion, and that the spring force is represented by $F_s$, where $F_s = -kx^2$ for the coordinate $x$ equal to the actual amount of contraction of the spring in inches. We shall restrict ourselves to motions where the spring is under contraction. The $x$-axis is assumed to be pointed downward. We shall suppose also that when motion begins, the friction force, opposing the motion, increases to a maximum absolute value $M$, dropping in absolute value to $K$ the instant motion starts. We suppose further that the friction force is zero when the body passes through $x' = 0$ only momentarily.

The assumptions made here concerning *dry friction* are satisfactory for many applications. See Section 1.13.

Let us suppose that a sinusoidal force represented by $F$ is applied to the weight, that is,

$$F = A \sin \omega t \tag{2.37}$$

for positive numbers $A$ and $\omega$. Starting with $t = 0$ it is desired to describe the resulting motion of the body $B$.

To set up the problem in mathematical form it is necessary to know the initial friction force. At the start this force is probably different from zero. To obtain this force we measure the length of the spring.

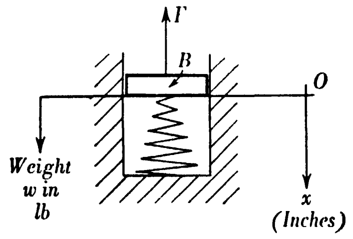

**Fig. 20. Sliding weight on a nonlinear spring.**

From this length the spring force can be computed. Subtracting the magnitude of this force from the weight $w$ of $B$, we have the magnitude of the initial friction force. Naturally, the *initial position of the body B is not unique.* The force of friction balances the effects of the weight $w$ and the spring force, provided that $F_c$ does not have to exceed $M$ in absolute value to do this.

We shall suppose that the initial value of $F_c$ is $-N$ where $N > 0$ and $N \leqq M$, and assume that $N < w$. The initial value $x_0$ of $x$ is such that

$$w - N + F_s = 0, \tag{2.38}$$

whence

$$x_0 = \sqrt{\frac{w - N}{k}}. \tag{2.39}$$

The reader is referred to Figure 21.

The body $B$ remains stationary as long as $-F_c$ given by the equation of equilibrium

$$F_c + F + F_s + w = 0 \tag{2.40}$$

does not exceed $M$. Equations (2.38) and (2.40) yield

$$-F_c = A \sin \omega t + N, \tag{2.41}$$

which implies that $B$ is stationary from $t = 0$ until

$$\sin \omega t = \frac{M - N}{A} \tag{2.42}$$

If $A$ is so small that $(M - N)/A$ is greater than or equal to 1, the body $B$ never moves.

When $(M - N)/A$ is less than 1, the body $B$ will move at the instant that $-F_c$ given by (2.41) exceeds $M$. Since $-F_c$ cannot physically exceed $M$, formula (2.41) no longer gives $-F_c$ when $B$ moves. When the body $B$ is on a downward stroke the relation

$$mx'' + kx^2 = w - K + A \sin \omega t. \tag{2.43}$$

applies. When $B$ is on an upward stroke, (2.43) applies with $K$ replaced by $-K$, in which case

$$mx'' + kx^2 = w + K + A \sin \omega t \tag{2.44}$$

It is thus seen that *it is sometimes necessary to treat different parts of a motion with different equations.*

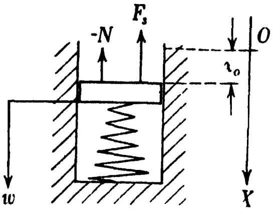

**Fig. 21. Sliding weight at the start of the motion.**

To obtain the motion of the body $B$ we should go through the following steps. As initial conditions take the smallest positive value of $t$ for which (2.42) is satisfied. We call this $t_0$. Solve (2.43) with the initial conditions

$$t = t_0, \quad x = x_0, \quad x' = 0, \tag{2.45}$$

obtaining a formula $f_1(t)$ for $x$. Find the smallest value $t_1$ of $t$ for which $t_1 > t_0$ and the derivative $f_1'(t)$ vanishes. Now solve (2.44) for $x$ subject to the initial conditions

$$t = t_1, \quad x = f_1(t_1), \quad x' = f_1'(t_1) = 0. \tag{2.46}$$

Let the solution of (2.44) and (2.46) be denoted by $f_2(t)$.

The process used to get $f_2(t)$ from $f_1(t)$ enables us to derive $f_3(t)$ from $f_2(t)$, interchanging, of course, the equations (2.43) and (2.44) in the argument. This sort of solution can be carried out for any number of steps. It follows that the *motion of the body $B$, once it starts, can be found by solving (2.43) and (2.44) successively with properly chosen initial conditions,* provided $x \geqq 0$ in the solution so that $F_s = -kx^2$ holds.

Where very rough results are enough, it is customary to replace the friction force used in this example by $-cx'$ for a properly chosen $c$, thus avoiding the method of successive solution used above. Where any accuracy is required the procedure outlined above or something equivalent must be employed. Various mathematical techniques can be used to shorten the work.

**2.9. Body attached to a spring hinged at one end (two-dimensional motion).** We consider the two-dimensional system of Figure 22 where the origin of our coordinate system is taken at the hinge. The coordinates of a point in the body $B$ are given by $(x, y)$. This point may be the center of gravity. The polar coordinates of this point are taken as $(\rho, \theta)$. For convenience we assume that this point is on the axis of the spring as shown. The symbol $\rho_0$ denotes the value of $\rho$ when the spring is neither stretched nor contracted. The spring force is given by $k(\rho - \rho_0)$. The horizontal and vertical components of this force are represented by $F_x$ and $F_y$, respectively. Allowing for signs it is seen from Figure 23 that

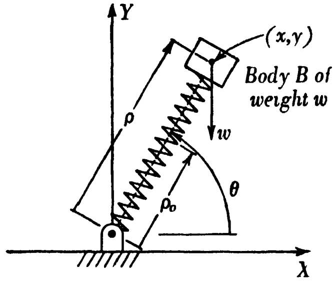

**Fig. 22. Body attached to a spring hinged at one end.**

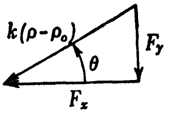

**Fig. 23. Components of a spring force.**

$$F_x = k(\rho_0 - \rho) \cos \theta, \quad F_y = k(\rho_0 - \rho) \sin \theta. \tag{2.47}$$

The total force acting on $B$ has the $x$-component represented by $F_x$ and the $y$-component represented by $(F_y - w)$ for the weight $w$ of $B$, whence the equations of motion are

$$\begin{aligned} mx'' &= k(\rho_0 - \rho) \cos \theta, \\ my'' &= k(\rho_0 - \rho) \sin \theta - w. \end{aligned} \tag{2.48}$$

The variables $\rho$, $\theta$ can be expressed in terms of $x$ and $y$, and eliminated from (2.48), giving equations in $x$ and $y$ only.

With the initial conditions

$$t = 0, \quad x = x_0, \quad x' = x_0', \quad y = y_0, \quad y' = y_0' \tag{2.49}$$

the motion of $B$ is completely described.

**2.10. Rocket moving in two-dimensional space.** There follows a case where the equations of motion may be very complicated if force components are taken in the directions of the coordinate axes.

We shall derive the equations of motion of a rocket traveling in a plane. The assumption is not valid for the long-distance flight of a missile like the German V-2 weapon, when the point of arrival must be known to great accuracy. For the latter case wind drift, the rotation of the earth, and other factors assume considerable importance.

We shall suppose that the rocket is lined up along its direction by fins so that the air resistance depends on the velocity only, and not on the attitude of the rocket with respect to the line of flight.

The rocket is propelled forward by a thrust $T$ due to the combustion of the fuel. This thrust will clearly be directed along the line of flight in the direction of motion as shown in Figure 24, where $m$ designates the mass of the rocket concentrated at its center of gravity. Our picture of the rocket as a particle is justified by the fact that it is practically always lined up along its path, so that knowing the path of the particle we also have the attitude of the rocket for any position of its center of gravity. By *attitude* we mean the direction of an axis fixed on the rocket. We are assuming that the rocket is symmetrical with respect to an axis along the line of flight, which is clearly satisfied by proper construction of the rocket.

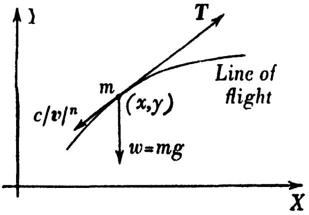

**Fig. 24. Rocket *m*.**

It is reasonable to suppose that the thrust $T$ is of constant magnitude until the fuel is exhausted when the thrust suddenly drops to zero.

The above assumptions were not sheer guesses. Anyone with enough experience in setting up differential equations would make these assumptions as the simplest ones that will reasonably fit the facts.

As practical experience shows, the air resistance may be taken to be of the form $c|\,\boldsymbol{v}\,|^n$ for numbers $c$ and $n$ depending on the altitude, and speed $|\,\boldsymbol{v}\,|$ of the rocket. The forces acting on the rocket are shown in Figure 24. The equations of motion are

$$\begin{aligned} mx'' &= (|\,\boldsymbol{T}\,| - c|\,\boldsymbol{v}\,|^n)\cos\alpha, \\ my'' &= (|\,\boldsymbol{T}\,| - c|\,\boldsymbol{v}\,|^n)\sin\alpha - w, \\ \alpha &= \tan^{-1}\frac{y'}{x'}, \quad |\,\boldsymbol{v}\,| = \sqrt{(x')^2 + (y')^2}, \end{aligned} \tag{2.50}$$

with the thrust $T$, angle $\alpha$ of inclination of the tangent ray to the line of flight in the direction of motion, and weight $w$ of the rocket. The primes denote derivatives with respect to time. When the thrust disappears we have (2.50) with $T$ equal to zero. For a given $c$ and a given $n$, equations (2.50) hold for a limited range of altitudes and speeds only, and must be changed from time to time during the flight, the initial conditions for one range being taken as the end condition of the last. To determine the flight we must know, for example, the initial position of the rocket, which may be taken to be the origin, and the initial velocity, that is, $x'$ and $y'$.

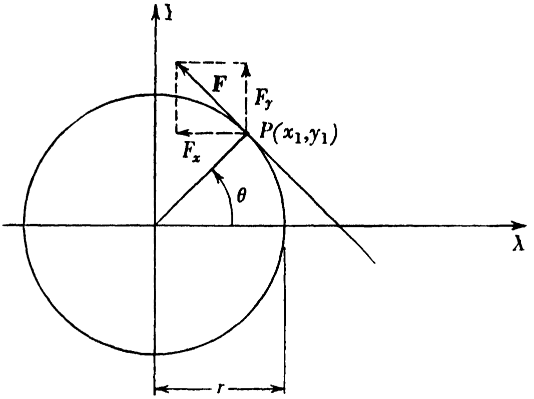

Fig. 25. Rotating particle.

In equations (2.50) we can express sin $\alpha$ and cos $\alpha$ in terms of $x'$ and $y'$. These equations are *nonlinear* differential equations of the *second order* in the variables $x$ and $y$.

**2.11. Rotation.** The type of *rotation* most important for engineering purposes is that of a rigid body about an axis. Each point in the body then describes a circle with center on the axis. In Figure 25 we have pictured a particle $P$ traveling on a given circle. Taking the center of the circle as the origin of a polar coordinate system, the position of the particle $P$ is determined by the vectorial angle $\theta$ of $P$. We shall use such an angle $\theta$ to designate the position of the given rigid body.

We superimpose a Cartesian coordinate system on the given polar coordinate system as shown in Figure 25. We let the radius of the given circle be denoted by $r$. The $(x, y)$-coordinates of $P$ are written as $(x_1, y_1)$. The relations between $x_1$, $y_1$, $r$, and $\theta$ are

$$x_1^2 + y_1^2 = r^2,$$
$$\tan\theta = \frac{y_1}{x_1}. \tag{2.51}$$

It will be no restriction on the generality of the argument to take the coordinate system so that $P$ is in the first quadrant at the instant considered. We can then write the second equation of (2.51) as

$$\theta = \tan^{-1}\frac{y_1}{x_1}, \tag{2.52}$$

the angle $\theta$ being now a single-valued function of $y_1/x_1$.

Since the derivatives of $\theta$ with respect to time are associated with the motion of $P$, it is reasonable to investigate $\theta''$, which by (2.52) and the first relation of (2.51) turns out to be such that

$$r^2\theta'' = x_1y_1'' - y_1x_1''. \tag{2.53}$$

Multiplying both sides of (2.53) by the mass $m$ of $P$, and using the fact that $F_x = mx_1''$, $F_y = my_1''$, we obtain

$$mr^2\theta'' = x_1F_y - y_1F_x \tag{2.54}$$

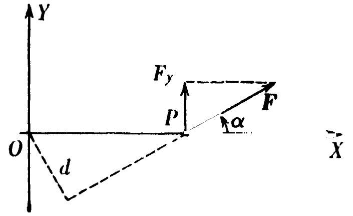

Fig. 26. Forces on a particle $P$.

relating $\theta''$ to the functions $F_x$ and $F_y$ representing $x$- and $y$-components of force.

In (2.54) the expression $mr^2$ *appears automatically*. Due to the similarity of (2.54) to (1.5) we are led to introduce a term and symbol for $mr^2$, namely the *moment of inertia* $I$ of the particle $P$ with respect to the axis of revolution. We do the same thing for the right side of (2.54), calling it the *torque* $M$, or *force moment* $M$ acting on $P$. Equation (2.54) is now

$$I\theta'' = M. \tag{2.55}$$

We note that the terms $x_1F_y$ and $-y_1F_x$ are positive when the forces corresponding to $F_y$ and $F_x$ tend to rotate $P$ in a counterclockwise direction.

It is no restriction on the generality of the method to take $P$ on the axis, in which case $y_1 = 0$. Equation (2.54) does not then involve $F_x$. We let $F$ designate the given force on $P$ as shown in Figure 26, and $d$ the distance from the line of this force to $O$. We let $\alpha$ be the angle from the positive direction on the $x$-axis to the ray of the force $F$. Since $F_y = |F| \sin\alpha$ we have

$$M = x_1|F|\sin\alpha. \tag{2.56}$$

Except for a ± sign, $d/x_1$ is the sine of $\alpha$. It follows that we can write (2.56) as

$$M = \pm d|F|, \tag{2.57}$$

the plus sign holding if sin $\alpha$ is positive, and negative otherwise. If sin $\alpha$ is positive the force $F$ is such as to tend to rotate $P$ counterclockwise, and if sin $\alpha$ is negative we have the opposite result. It follows that the *moment of a force* **F** *with respect to O is plus or minus the product of the magnitude of* **F** *by the distance from the line of* **F** *to the origin, the resulting moment being taken positive when* **F** *tends to accelerate P counterclockwise, and if clockwise, then negative.* We understand, by definition, that $F$ *tends* to rotate $P$ counterclockwise if the component represented by $F_y$ is directed upward, that is, $F_y$ is positive for $P$ as located in Figure 26.

The concept of moment, as defined, does not depend on a particular position of the particle $P$. It is easily seen that if several forces are acting on $P$, the *moment of the sum* of these forces *is the sum of the moments* of these forces. In fact, suppose that forces $F$ and $G$ are acting on $P$. Let $M$ be the torque due to the $y$-components of force represented by $F_y$ and $G_y$. By the additive property of forces $M$ is $x_1(F_y + G_y)$. Since

$$x_1(F_y + G_y) = x_1F_y + x_1G_y \tag{2.58}$$

the proposition is demonstrated.

By integration we can extend the concepts of moment of inertia $I$ and torque $M$ to rigid bodies. In particular, consider the body $B$ of Figure 4, and restrict it to rotation about the $z$-axis. We let $r_i$ be the distance from the point $P_i$ to the $z$-axis. We introduce the moment of inertia element $\Delta I_i$, where by definition

$$\Delta I_i = r_i^2\Delta m_i. \tag{2.59}$$

Here $\Delta m_i$ is the mass of the element with volume $\Delta V_i$, where $\Delta V_i = \Delta x_i\Delta y_i\Delta z_i$.

We introduce the element of torque $\Delta M_i$ for which

$$\Delta M_i = x_i\Delta F_{yi} - y_i\Delta F_{xi} \tag{2.60}$$

for the functions $\Delta F_{xi}$, $\Delta F_{yi}$ representing the $x$- and $y$-components of the force on the element of mass $\Delta m_i$.

For the angular acceleration $\theta''$ of the body $B$, equation (2.53) applies with the subscript $i$ on all of the variables except $\theta$, whence multiplying both sides by $\Delta m_i$, applying (2.59) and (2.60), we have

$$\theta''\Delta I_i = \Delta M_i. \tag{2.61}$$

We subdivide the region occupied by $B$, using $n$ nonoverlapping parallelepipeds with faces parallel to the axes, where each point of $B$ is in or on at least one parallelepiped. We let $\delta$ be the maximum length of the chords joining points of a parallelepiped. We define the moment of inertia $I$ of the body $B$ with respect to the $z$-axis by the limit

$$I = \lim_{\delta \to 0} \sum_{i=1}^{n} \Delta I_i. \tag{2.62}$$

We define the total torque by the same limit process in terms of the $\Delta M_i$'s. By integration it follows from (2.61) that (2.55) holds for the body $B$ so that we have proved the following important theorem.

***Theorem* 2.1.** *For a body $B$ rotating about an axis the applied torque $M$ is the product of the moment of inertia $I$ of $B$ by the angular acceleration $\theta''$, the quantities $M$, $I$, and $\theta''$ being taken relative to the axis of rotation.*

Since an angle $\theta$ in radians is an arc of a circle divided by the radius of the circle, $\theta$ being a central angle, it follows that $\theta$ has the dimensionality $L/L$ or 1. Thus time derivatives $\theta'$ and $\theta''$ have the *dimensionality* $1/T$ and $1/T^2$ respectively. By (2.55) the dimensionality of $I$ is the dimensionality $FL$ of the torque $M$ over the dimensionality $1/T^2$ of $\theta''$, whence $I$ has the dimensionality $FLT^2$.

The moment of inertia $I$ is often in inch-pound seconds$^2$ (in lb sec$^2$), or *foot-pound seconds*$^2$ (ft lb sec$^2$). Torque is usually given in inch-pounds (in lb) or *foot-pounds* (ft lb).

Because of the similarity of the force and torque laws (1.5) and (2.55), we would expect results obtained from (1.5) to also apply to (2.55). It is easy to prove that, for example, the kinetic energy of a body rotating about an axis with angular velocity $\theta'$ is $\frac{1}{2}I(\theta')^2$ for the moment of inertia $I$ of the body.

Principle 1.1 can be extended to torques, and implies, for the conditions stated in this principle, that *torques are functions* of $\theta$ and $\theta'$ when rotation about an axis is being studied.

It is a simple matter to show that the formula (1.26) applied to rotations about axes becomes

$$W = \int_a^\theta M d\theta \tag{2.63}$$

for the initial angle $a$ and the final angle $\theta$.

### 2.12. Throttle movement and speed of a diesel engine.

We consider a diesel engine where the speed of the crankshaft is denoted by $N$, taken in revolutions per minute (rpm), and the "throttle" coordinate of position (rack position) is designated by $x$. See Figure 27 for a photograph of a cross section of a diesel engine.

Let $N_1$ be the speed of the crankshaft in radians per second. The torque applied to the crankshaft by the gases is controlled by the throttle, and will therefore be called the "*throttle torque*." This torque oscillates because of the intermittent character of the explosions. Since the explosions occur frequently, it is a good approximation to actual experience to work with an average value, and to assume that this average value is directly proportional to $x$, that is, is $kx$. It is convenient to take $x$ in inches. We have *chosen the diesel engine* for this example because of its rapid throttle torque response to throttle movements.

The throttle torque $kx$ is opposed by the *inertial torque* $-IN_1'$, where $I$ is the moment of inertia of the crankshaft and whatever is connected to it, relative to the axis of the shaft, and a damping torque $f(N_1)$ due to friction in the system. Experiments show that $f(N_1)$ is approximately $-cN_1$ for a number $c$. By equation (2.55) we have

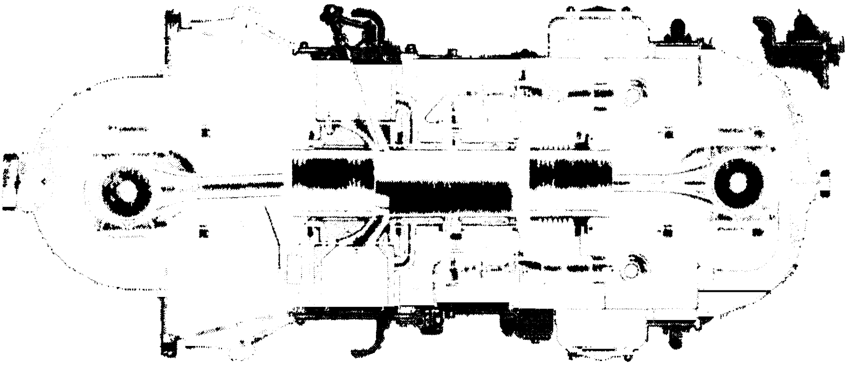

**Fig. 27. Cross section of a diesel engine with opposed pistons. Courtesy of the Fairbanks, Morse & Company.**

$$IN_1' + cN_1 = kx. \tag{2.64}$$

It is common practice to take the torques in (2.64) in foot-pounds. With the initial condition

$$t = 0,\ N_1 = N_{10}$$

and $x$ given as a function of $t$, the motion is determined by (2.64).

In engineering practice, at each cylinder one-half the mass of the piston and most of the mass of the connecting rod joining this piston to the crankshaft are placed at the junction of the connecting rod and crankshaft in computing the contribution of these parts to the moment of inertia $I$. This method of accounting for the reciprocating parts of a reciprocating engine can be justified by the fact that we are then using approximately average effects of the piston and connecting rod mass on the rotation of the crankshaft.

For a diesel engine driving a generator not coupled with others, (2.64) still applies, provided the inertia of the generator rotor and the friction in the generator (and windage) are included in $I$ and $c$, respectively.

When we use revolutions $N$ per minute instead of $N_1$, equation (2.64) becomes

$$\frac{\pi}{30}(IN' + cN) = kx. \tag{2.65}$$

This equation is preferred in practice to (2.64).

A typical numerical example is one for which $I = 100$, $c = 10$, $k = 20{,}000$. We are taking $t$ in seconds, and $x$ in inches. Suppose that the engine is running at 1,000 revolutions per minute, and the throttle is suddenly closed. Assuming that the throttle is closed at $t = 0$, we have $N = 1{,}000$ for $t < 0$. For $t \geqq 0$ we have $x \equiv 0$, whence the differential equation

$$10N' + N = 0 \tag{2.66}$$

applies. Since *the engine cannot change its speed instantaneously*, we have the initial condition

$$t = 0,\ N = 1{,}000. \tag{2.67}$$

Solution of (2.66) and (2.67) yields

$$N = 1{,}000e^{-\frac{t}{10}}. \tag{2.68}$$

The method just used for obtaining initial conditions for a sudden disturbance is typical.

The coefficient of the $N$ term in (2.66) is small compared to that of the $N'$ term. This suggests dropping the $N$ term in (2.66). Solution of this equation then gives

$$N = 1{,}000. \tag{2.69}$$

If we expand the power of $e$ in (2.68) into a power series, the equation (2.68) assumes the form

$$N = 1{,}000\left(1 - \frac{t}{10} + \frac{t^2}{200} - \cdots\right). \tag{2.70}$$

By the theory of *alternating infinite series* the error made by dropping all but the first term in the series is no greater than the absolute value of the first term neglected. Thus the use of (2.69) instead of (2.70) causes an error $\Delta N$ in $N$ such that

$$\Delta N \leqq 100t. \tag{2.71}$$

By (2.71) the solution (2.69) is correct to 10 per cent for the first second. For practical purposes this may be good enough. We would then say that for the first second the speed $N$ is "constant."

It is easily seen that Principle 2.1 applies to torques as well as forces where *static equilibrium* is replaced by the *equilibrium for which the angular speed is constant.* We shall apply this principle in the example of the present section, but for the disturbance of instantly moving the throttle from the position which corresponds to an equilibrium speed of 1,000 revolutions per minute to that which corresponds to 1,500 revolutions per minute. This disturbance is not as severe as that treated above. We shall measure our torques from the initial equilibrium state.

By (2.65) the coordinate $x$ jumps suddenly from $x = 0$ representing equilibrium at 1,000 revolutions per minute to $x = \pi/120$ corresponding to a jump from $N = 0$ (representing 1,000 revolutions per minute) to $N = 500$ (representing 1,500 revolutions per minute). The mathematical system is

$$\begin{aligned} t &= 0, & N &= 0, \\ t &> 0, & 10N' + N &= 500. \end{aligned} \tag{2.72}$$

Solution of (2.72) yields

$$N = 500(1 - e^{-\frac{t}{10}}). \tag{2.73}$$

Let us drop the $N$ term in the engine-throttle equation of (2.72). The solution is now

$$N = 50t. \tag{2.74}$$

If we expand the power of $e$ in (2.73) into a power series in $t$ we obtain

$$N = 50t - 2.5t^2 + \dots . \tag{2.75}$$

Dropping the $N$ term in the engine-throttle equation of (2.72) thus corresponds to keeping only the first term of (2.75). The *error* in using (2.74) as the solution of $N$ instead of (2.75) is less than 5 per cent of $N$ ($N$ being measured as deviation from 1,000 revolutions per minute) for the first second, less than 10 per cent for the first two, and less than 20 per cent for the first four. For engineering purposes an error of even 20 per cent is sometimes satisfactory, the error in measurements of transients in this type of problem often being of this order.

The success attained in throwing away the $N$ term in the engine-throttle equation of (2.72) (and this means throwing away the damping term) depends on the magnitude of the $N'$ term compared with the $N$ term at $t = 0$. In fact, at $t = 0$, the first term of the engine-throttle equation is 500 and the second is zero. Since $N$ and $N'$ are continuous functions of $t$, we would expect that for quite some time the second term would be small compared to the first, and this is exactly what happens.

The argument of the preceding paragraph will be emphasized by taking the reference equilibrium state in the last example to be that for which $N = 1{,}500$ revolutions per minute instead of 1,000 revolutions per minute. Measuring our variables from this state we have the mathematical system

$$\begin{aligned} t &= 0, & N &= -500, \\ t &> 0, & 10N' + N &= 0. \end{aligned} \tag{2.76}$$

At $t = 0$, the $N'$ term and the $N$ term are of the same magnitude. Dropping the $N$ term gives the solution

$$N = -500, \tag{2.77}$$

which is a misrepresentation of what actually happens.

In general, only by considerable experience with mathematical equations and experimental results can one tell quickly whether or not low-order terms can be dropped from differential equations, and corresponding physical factors neglected. It is readily seen that *high-order terms with relatively small coefficients can generally be dropped from a differential equation.*

**2.13. Diesel engine driving a ship propeller.** We shall treat the case of a marine diesel engine where the rotating part of the engine is rigidly connected to a ship propeller. Here *damping is important.* We let $I$ be the moment of inertia of the rotating part of the diesel engine, propeller, and connecting shaft. The water resists the propeller with a torque which increases with the speed of the engine. This torque is very nearly of the form $\pi \alpha N^n/30$ for constants $\alpha$ and $n$, and engine revolutions per minute $N$. With throttle torque written in the general form $\pi f(x)/30$, the differential equation of motion is

$$IN' + \alpha N^n = f(x). \tag{2.78}$$

In (2.78) the values to be taken for $n$ and $\alpha$ depend on the speeds at which the propeller is driven. A common value of $n$ is 2.5. The higher the speed, the higher $n$ must be.

Let us suppose that in an equilibrium state we have

$$x = x_0, \quad N = N_0. \tag{2.79}$$

By (2.78) we have $\alpha N_0^n = f(x_0)$, whence

$$I(\Delta N)' + \Delta(\alpha N^n) = \Delta f, \tag{2.80}$$

where

$$\Delta N = N - N_0, \quad \Delta(\alpha N^n) = \alpha(N^n - N_0^n), \quad \Delta f = f(x) - f(x_0). \tag{2.81}$$

By the calculus we have (assuming the validity of the mathematical operations)

$$\Delta N \approx dN, \quad \Delta x \approx dx, \quad \Delta(\alpha N^n) \approx n\alpha N_0^{n-1} dN, \quad \Delta f = f'(x_0)dx, \tag{2.82}$$

where $\approx$ means "*is approximately equal to.*" Introducing the constants $c$ and $k$ by the relations

$$c = n\alpha N_0^{n-1}, \quad \frac{30k}{\pi} = f'(x_0), \tag{2.83}$$

and replacing $\Delta N$ by $N$ and $\Delta x$ by $x$, the equation of motion (2.80) becomes (2.65).

At propeller speeds like 100 revolutions per minute the damping coefficient $c$ may be just as large as $I$.

The above example illustrates *how equations are linearized* by the use of differentials.

### 2.14. Gas turbine driving a generator.

Both damping and position forces occur in this case.

We shall suppose that there is an angle $\theta$ of synchronization such that if $\theta = 0$, the given generator is synchronous with others on the line, it being assumed that the generator is tied in with many others of the same or greater capacity. The angle $\theta$ is an angle between a ray passing through and perpendicular to the axis of the generator rotor and through a fixed point on the rotor off the axis, and a reference ray rotating about the axis of revolution so that at each instant this ray indicates the synchronous position.

There is a synchronizing torque which increases with $\theta$. For practical purposes we may take this torque to be $-k\theta$ for a number $k$. A damper winding in the generator provides a damping torque of the form $-c\theta'$ for a number $c$. Letting $x$ be measured from some equilibrium position, we write the throttle torque as $k_1 x$ for a number $k_1$. We assume instant torque response of the turbine to throttle movements. The equation of motion is

$$I\theta'' + c\theta' + k\theta = k_1 x \tag{2.84}$$

for the moment of inertia $I$ of the rotating member.

In a typical example $c = \frac{1}{2}I$ and $k = 300I$.

For (2.84) we really did not need to know the type of engine driving the generator.

### 2.15. Shaft connected to a rotating body.

We shall consider Figure 28. Here one end of a shaft is rotating according to a given law. It may, for example, be turned by an engine. The body $B$ may be a propeller for which damping can be neglected in this problem, an engine flywheel, the rotor of a gas turbine air compressor, the rear wheel of an automobile when not touching the ground, or other bodies too numerous to mention. A system mathematically equivalent to that of Figure 28 is shown in Figure 29. The machinery may be a milling machine (not the case in Figure 29), in fact any device for which the inertias of the moving parts can be referred to the right-hand pulley of Figure 29. Let $I$ designate the moment of inertia of the rotating parts.

In Figure 28 the symbols $\psi$ and $\theta$ refer to angular positions of the left end of the shaft and the body $B$, respectively. We shall suppose that $\psi$ and $\theta$ are measured so that the shaft is not twisted with $\psi = \theta$. The *torsional stiffness* $k$ is by definition such that a torque $k\theta$ is required to twist the right end of the shaft through an angle $\theta$ when the left end is fixed. The stiffness $k$ is the angular analogue of spring scale. The equation of motion is

$$I\theta'' = k(\psi - \theta), \tag{2.85}$$

which is better recognized in the form

$$I\theta'' + k\theta = k\psi. \tag{2.86}$$

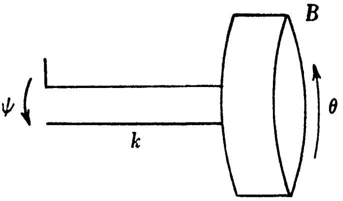

Fig. 28. Shaft driving a mass.

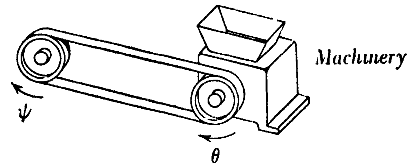

Fig. 29. Machinery driven from a belt.

If we solve the homogeneous equation obtained from (2.86) by setting $\psi$ equal to zero, we obtain an oscillatory solution with the frequency $f$, where

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{I}}. \tag{2.87}$$

This $f$ is the natural frequency of the system.

In a typical marine shaft-propeller example $I = 20{,}000$, and $k = 10^7$, the units being pounds, inches, seconds, and radians.

### 2.16. Diesel engine driving a marine propeller through a long shaft.

We shall consider a more complicated example involving stiffness.

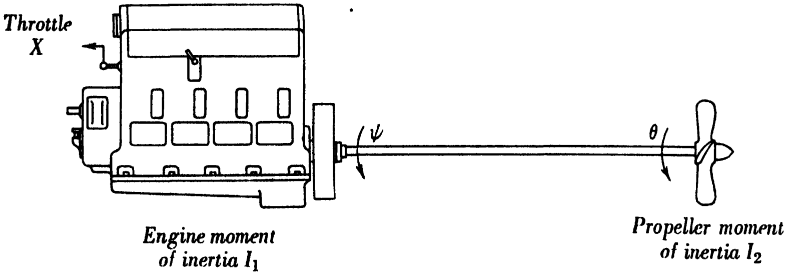

Fig. 30. Diesel engine driving a marine propeller through a long shaft.

In Figure 30 the variables $\psi$ and $\theta$ are angular positions as in the example of Section 2.15. The other notations are explained by the figure. The equations of motion are now

$$\begin{aligned} I_1\psi'' &= k_1x + k(\theta - \psi), \\ I_2\theta'' &= -c\theta' + k(\psi - \theta) \end{aligned} \tag{2.88}$$

for the constants $I_1$, $I_2$, $k_1$, $k$, and $c$. The set of initial conditions

$$t = 0,\quad \theta = \theta_0,\quad \theta' = \theta'_0,\quad \psi = \psi_0,\quad \psi = \psi'_0, \tag{2.89}$$

and (2.88) determine the motion of the parts of the system.

See Figure 31 for an example of a marine shaft and propeller.

Fig. 31. A marine shaft and propeller. Courtesy of S. Morgan Smith Company.

**2.17. Airplane.** We shall treat a problem where the body is rotating about a moving axis. This body will be the airplane of Figure 32, where $w$ is the weight of the plane, shown acting at the center of gravity $(x, y)$. In this figure the symbol $D$ represents the total component of force acting along the line of flight, leaving gravity out of the picture. Thus $D$ comes from the drag and propeller thrust. Similarly, $L$ represents the total component of force acting perpendicular to the line of flight, without gravity. Of course, the thrust may come from some other propulsion devices such as jet engines or rockets. The forces represented by $L$ and $D$ are taken to act at a point, as shown.

The angle $\phi$ in Figure 32 is measured from the horizontal to a fixed axis in the plane through the center of gravity. The angle the airplane velocity vector $v$ makes with the horizontal is denoted by $\theta$.

We let $-M$ designate the moment of the forces, represented by $L$ and $D$, with respect to the center of gravity. With $m$ as the mass of the airplane and $I$ as moment of inertia we have the equations of motion

$$\begin{aligned} I\phi'' &= -M, \\ mx'' &= -L\sin\theta - D\cos\theta, \\ my'' &= L\cos\theta - D\sin\theta - w. \end{aligned} \tag{2.90}$$

These equations correspond to the three degrees of freedom $x$, $y$, and $\phi$.

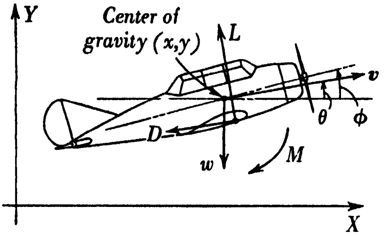

Fig. 32. Airplane in motion.

**2.18. Conservative forces and potential energy.** Let us suppose that the force represented by $F$ is *conservative*, that is, the work $W$ defined by (1.26) in moving the particle from one point to a second point depends only on these points, and not on the manner in which the particle went from one point to the other. We note that spring forces represented by functions $\{F(x)\}$ are conservative, since in this case

$$\int_a^b F(x)dx = f(b) - f(a) \tag{2.91}$$

for a function $f(x)$ whose derivative is $F(x)$. *Damping forces are not conservative.* Thus the integral

$$\int_a^b F(x')dx$$

does not depend on the end points only. We can move a piston in a dashpot from one position to another through all sorts of oscillations that require different amounts of work by the piston against the damping force. No matter how the piston is moved work must always be done against the damping force.

When the work $W$ depends on the end points only, we can associate a potential energy $U$ with the particle so that $W$ is the potential energy at the start minus this energy at the end. Writing $W$ as $(A - U)$ for a number $A$, arbitrarily chosen, relation (1.29) yields

$$T + U = K \tag{2.92}$$

for a constant $K$. Equation (2.92) is an example of the law of the conservation of energy. In actual practice the relation

$$T_1 + U_1 = T_2 + U_2 \tag{2.93}$$

free of $K$ is commonly used, where the subscripts indicate energies in two respective states. The above considerations extend immediately to systems of particles and continuous mass distributions.

If $F(x)$ in (2.91) is a constant, for example $F_0$, we have $W = F_0(b - a)$ for the work done in going from $x = a$ to $x = b$. In this case, the particle possesses a potential energy proportional to the distance between initial and final points. Thus a body of weight $w$ raised a distance $x$ against the force of gravity near the earth has a potential energy $wx$.

A spring compressed a distance $x$ can do the work

$$\int_0^x kx\,dx, \tag{2.94}$$

where $k$ is the scale of the spring. The potential energy possessed by a spring compressed a distance $x$ is thus $\frac{1}{2}kx^2$.

We consider the body $B$ of Figure 11. We shall assume, as is customary in treating vibrations, that the body $B$ moves according to the law

$$x = A \sin \omega t \tag{2.95}$$

for numbers $A$ and $\omega$. That this treatment is quite general follows from the theory of Fourier series where we break up a rather general periodic function into a sum of terms in sines and cosines.

When $x = A$ the kinetic energy $T_1$ of the body $B$ is zero, while the potential energy $U_1$ of the spring is $\frac{1}{2}kA^2$ for the spring scale $k$. When $x = 0$ the velocity of the body $B$ is $\pm\omega A$ by (2.95), whence the kinetic energy $T_2$ of the body $B$ is $\frac{1}{2}m\omega^2A^2$ for the mass $m$ of $B$. The potential energy $U_2$ of the spring is then zero. By (2.93)

$$\tfrac{1}{2}kA^2 = \tfrac{1}{2}m\omega^2A^2. \tag{2.96}$$

Solution of (2.96) gives $\omega = \sqrt{k/m}$.

Thus this *energy method* yields a property of the solution of the differential equation of motion, even though it does not give the solution. A generalized method of Rayleigh and Ritz is very successful in treating physical systems with an infinite number of degrees of freedom. An example is the determination of the frequencies of vibrating beams.

Extensive mathematical theories can be built from equation (1.5), such as Lagrange's equations of motion for a general coordinate system, where these theories are useful for setting up physical systems in mathematical form. For one-dimensional motion under conservative forces the law of conservation of energy yields the equation of motion immediately.

Consider the body $B$ of Figure 11 at any position in its motion, the position being denoted by the coordinate $x$ as in Figures 11 and 12. The kinetic energy of $B$ is $\frac{1}{2}m(x')^2$, and the potential energy of the spring is $\frac{1}{2}kx^2$. By (2.92)

$$m(x')^2 + kx^2 = K \tag{2.97}$$

for a constant $K$.

If we integrate (2.5), after multiplying both sides by $x'$, we obtain (2.97). Thus *the equation of motion (2.97) obtained from the law of the conservation of energy is an integrated form of the ordinary equation of motion* (2.5) obtained directly from (1.5).

The intelligent worker will use tricks such as the above in solving applied problems, the techniques varying with the problem at hand. To be able to do this efficiently he must have considerable experience and knowledge of advanced mathematics as well as engineering.

The *potential energy* stored in a shaft due to the spring effect is $\frac{1}{2}k\theta^2$ for the torsional stiffness $k$ and angle of twist $\theta$. This is quite analogous to the case of a linear spring.

**2.19. Forces that can be neglected.** In any engineering application account must be taken of the fact that a *spring* never furnishes a pure spring force, but is associated with inertial forces due to its mass, and with damping due to friction in the spring, which friction can be recognized by heat developed in the spring. However, if the forces for a spring are represented by $-mx''$, $-cx'$, and $-kx$ for a position coordinate $x$, it is common to have $m$ and $c$ as small as $10^{-5}k$ and $10^{-3}k$, respectively, so that in many applications the damping and mass in a spring may be neglected. Of course, if the velocity and acceleration are very large compared to the displacement, such as $10^6$ times the maximum value of the displacement coordinate, these other forces may have to be taken into consideration.

Even for a poor spring, $m$ ordinarily will not exceed $10^{-3}k$, and $c$ will be negligible.

In a *dashpot* the piston has a mass $m$. It is common to have $m$ as small as $10^{-4}c$, and of course, no spring effect, the damping coefficient being $c$. Naturally, by using a heavy piston or attachment to the piston the mass $m$ can be made very large compared to $c$.

For all kinds of *rigid* bodies the damping within the body and the elasticity can be neglected, in fact in many applications these are completely negligible. They are *negligible because of the smallness of the motion of the parts of the body relative to each other, and not because spring and damping coefficients are small.*

We have enumerated the above characteristics of springs, dashpots, and rigid bodies because a machine made from mechanical parts can often be analyzed into such elements. It is therefore important in the mathematical analyses of machinery to be familiar with the common characteristics of these elements.

Experience in situations similar to the one being studied is of considerable help in deciding what forces may be neglected, and what types of dominant forces to expect.

**2.20. Primary and secondary quantities.** The fundamental equations of motion of the physical world are equations relating variables associated with various dimensions, and dimensional constants. The constant $k$ of (1.9) and (1.12) is an example of a dimensional constant. We are often concerned with obtaining an equation relating variables of physical systems of a particular type. It may be too difficult, for practical reasons, to derive this equation from the fundamental equations of motion, in which case resort is often made to dimensional analysis.

A variable is called *primary* if its value is obtained by measurement in terms of a unit. Length and time are examples of quantities generally taken as primary. *Secondary* quantities are obtained by mathematically combining measurements on primary quantities.

Let $u_1, u_2, \ldots, u_m$ be a set of primary quantities, and $f(u_1, u_2, \ldots, u_m)$ a secondary quantity. Let $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_m)$ be two sets of values of $(u_1, u_2, \ldots, u_m)$. If we make the unit of measure for the first primary quantity $1/x$ times as large, the quantity $u_1$ goes into $xu_1$. Under this change of a unit of measure the ratio of any two values of the secondary quantity should remain invariant. This means that

$$\frac{f(a_1, a_2, \ldots, a_m)}{f(b_1, b_2, \ldots, b_m)} = \frac{f(xa_1, a_2, \ldots, a_m)}{f(xb_1, b_2, \ldots, b_m)}$$

for all real values of $x$, the $a$'s, and $b$'s. We can write this as

$$f(xa_1, a_2, \ldots, a_m) = f(xb_1, b_2, \ldots, b_m)\frac{f(a_1, a_2, \ldots, a_m)}{f(b_1, b_2, \ldots, b_m)}.$$

We assume that the first partial derivatives of $f$ with respect to $u_1, u_2, \ldots, u_m$ exist for all real values of the $u$'s. Differentiating both sides of the last equation with respect to $x$ and setting $x = 1$ we have

$$a_1\frac{\partial f(a)}{\partial a_1} = b_1\frac{\partial f(b)}{\partial b_1}\frac{f(a)}{f(b)},$$

where $f(a)$ and $f(b)$ designate $f(a_1, a_2, \ldots, a_m)$ and $f(b_1, b_2, \ldots, b_m)$ respectively. Thus

$$\frac{a_1\dfrac{\partial f(a)}{\partial a_1}}{f(a)} = \frac{b_1\dfrac{\partial f(b)}{\partial b_1}}{f(b)} \tag{2.98}$$

is an identity in the $a$'s and $b$'s. Fix the $b$'s. Relation (2.98) shows that

$$\frac{u_1}{f(u)}\frac{\partial f(u)}{\partial u_1} = \alpha_1$$

for a constant $\alpha_1$, where we have replaced the $a$'s by $u$'s. Separating variables we have

$$\frac{\partial f(u)}{f(u)} = \alpha_1 \frac{\partial u_1}{u_1}. \tag{2.99}$$

Integrating both sides of (2.99) we obtain

$$\ln f = \alpha_1(\ln u_1 + K),$$

where we shall write $K$ as

$$K = \frac{\ln g}{\alpha_1}$$

for a function $g$ of $u_2, u_3, \ldots, u_m$. The solution of (2.99) is now

$$f = gu_1^{\alpha_1}.$$

By the argument applied to $f$ we can show (if $m \geqq 2$) that

$$g = hu_2^{\alpha_2}$$

for a function $h$ of $u_3, u_4, \ldots, u_m$ and a constant $\alpha_2$. Following this line of argument we find that

$$f = Cu_1^{\alpha_1}u_2^{\alpha_2} \cdots u_m^{\alpha_m}$$

for constants $C, \alpha_1, \alpha_2, \ldots, \alpha_m$. Thus a *secondary quantity is a constant times a product of powers of primary quantities.* If the dimensions of a physical system are $L_1, L_2, \ldots, L_m$, each secondary quantity has dimensionality of the form

$$L_1^{\alpha_1}L_2^{\alpha_2} \cdots L_m^{\alpha_m}, \tag{2.100}$$

where the exponents are the *dimensions* of the secondary quantity in the corresponding primary quantities. The dimensional formula for this secondary quantity is $(\alpha_1, \alpha_2, \ldots, \alpha_m)$, but this formula is generally replaced by (2.100).

A table of dimensions of various physical quantities is given in an appendix at the end of this book.

**2.21.** ***The Buckingham Pi Theorem.*** A *complete equation* is one which is invariant when the sizes of the fundamental units are changed, that is, the equation does not change in form for this transformation of the fundamental units. Equation (1.8) is an example of a complete equation, for it does not change in form when the units of mass, length, and time are modified. In industry, we are generally concerned with complete equations.

***Theorem 2.2.*** **(Buckingham Pi Theorem).** *Let*

$$f(v_1, v_2, \ldots, v_n) = 0 \tag{2.101}$$

*be a complete equation in the dimensional variables and constants $v_1, v_2, \ldots, v_n$, where these variables and constants are dimensionally products of powers of* *m primary quantities. Equation (2.101) can be written for a function F as*

$$F(\pi_1, \pi_2, \ldots, \pi_\sigma) = 0 \tag{2.102}$$

*where* $\pi_1, \pi_2, \ldots, \pi_\sigma$ *are dimensionless quantities, and* $\sigma \geqq n - m$.

For each $i$ in the range $1, 2, \ldots, n$ we let $\alpha_i$ be the dimension of $v_i$ in the first primary quantity. We change the size of the unit of measure of the first primary quantity by a factor $1/x$, so that for each $i$ the quantity $v_i$ goes into $x^{\alpha_i}v_i$. Since equation (2.101) is complete, the relation

$$f(x^{\alpha_1}v_1, x^{\alpha_2}v_2, \ldots, x^{\alpha_n}v_n) = 0 \tag{2.103}$$

is an identity in $x$.

We may suppose, without restriction on the generality of the argument, that for some $r$ with $r \leqq n$ we have $\alpha_1 = \alpha_2 = \ldots = \alpha_{r-1} = 0$, and $\alpha_r, \alpha_{r+1}, \ldots, \alpha_n$ different from zero. When $v_n \neq 0$ we write

$$x = \frac{1}{v_n^{\frac{1}{\alpha_n}}}. \tag{2.104}$$

For $i$ in the range $r, r + 1, \ldots, n - 1$ we introduce $z_i$, where

$$z_i = \frac{v_i^{\frac{1}{\alpha_i}}}{v_n^{\frac{1}{\alpha_n}}}. \tag{2.105}$$

By (2.104) and (2.105) the identity (2.103) becomes

$$f(v_1, \ldots, v_{r-1}, z_r^{\alpha_r}, \ldots, z_{n-1}^{\alpha_{n-1}}, 1) = 0,$$

and equation (2.101) reduces to an equation

$$g(v_1, \ldots, v_{r-1}, z_r, \ldots, z_{n-1}) = 0 \tag{2.106}$$

in the $n - 1$ quantities $v_1, \ldots, v_{r-1}, z_r, \ldots, z_{n-1}$, which quantities are dimensionless in the first primary quantity.

Unless the $n - 1$ variables in (2.106) are already dimensionless we can apply the above argument to equation (2.106) instead of equation (2.101), using another primary quantity. If necessary we can continue this process until the primary quantities are exhausted. In any event we arrive at an equation of type (2.102).

The transformations used to obtain equation (2.102) involve only the mathematical operations of raising to a power, and taking a ratio. In view of these operations the $\pi$'s are dimensional quantities. The proof, however, shows that the $\pi$'s are dimensionless in the $m$ primary quantities.

The assumption that $v_n$ be different from zero is no real restriction on the generality of the argument, since the analyst will certainly be interested in this case. If the $v$'s and $\alpha$'s in (2.105) have certain properly chosen values, the variable $z_i$ will take on an imaginary value. The theory above, however, applies whether we are working with real or complex numbers.

For each $i$ in the range $1, 2, \ldots, n$ we now write the dimensionality of $v_i$ as $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{im})$, or in the more common notation as

$$L_1^{\alpha_{i1}} L_2^{\alpha_{i2}} \ldots L_m^{\alpha_{im}}.$$

A $\pi$-term is of the form

$$\pi = v_1^{a_1} v_2^{a_2} \ldots v_n^{a_n}.$$

From the dimensionality of the $v$'s we can represent the dimensionality of the $\pi$-term by

$$(L_1^{\alpha_{11}} \ldots L_m^{\alpha_{1m}})^{a_1} (L_1^{\alpha_{21}} \ldots L_m^{\alpha_{2m}})^{a_2} \ldots (L_1^{\alpha_{n1}} \ldots L_m^{\alpha_{nm}})^{a_n}.$$

Since $\pi$ is dimensionless the exponent of each $L$ term must vanish, so that

$$\begin{aligned}
\alpha_{11}a_1 + \alpha_{21}a_2 + \ldots + \alpha_{n1}a_n &= 0,\\
\alpha_{12}a_1 + \alpha_{22}a_2 + \ldots + \alpha_{n2}a_n &= 0,\\
\cdot \quad \cdot \quad \cdot & \\
\alpha_{1m}a_1 + \alpha_{2m}a_2 + \ldots + \alpha_{nm}a_n &= 0.
\end{aligned}$$

We have thus $m$ equations in the $n$ unknowns $a_1, a_2, \ldots, a_n$. Generally $n > m$, and we can usually solve for $m$ of the $a$'s in terms of the remaining $n - m$. There are exceptions, as follows from the theory of linear equations. In any case, we can find the $\pi$-terms by solving a system of linear equations.

Suppose that we can solve the above equations for $a_1, a_2, \ldots, a_m$ in terms of $a_{m+1}, a_{m+2}, \ldots, a_n$. For properly chosen values of the $b$'s we have

$$\begin{aligned}
a_1 &= b_{1,m+1}a_{m+1} + b_{1,m+2}a_{m+2} + \ldots + b_{1n}a_n,\\
a_2 &= b_{2,m+1}a_{m+1} + b_{2,m+2}a_{m+2} + \ldots + b_{2n}a_n,\\
& \cdot \quad \cdot \quad \cdot \\
a_m &= b_{m,m+1}a_{m+1} + b_{m,m+2}a_{m+2} + \ldots + b_{mn}a_n.
\end{aligned}$$

The general solution for $\pi$ can now be written as

$$\pi = \pi_1^{a_{m+1}} \pi_2^{a_{m+2}} \ldots \pi_{n-m}^{a_n}$$

for the $\pi$-terms $\pi_1, \pi_2, \ldots, \pi_{n-m}$ given by

$$\begin{aligned}
\pi_1 &= v_1^{b_{1,m+1}} v_2^{b_{2,m+1}} \ldots v_m^{b_{m,m+1}} v_{m+1},\\
\pi_2 &= v_1^{b_{1,m+2}} v_2^{b_{2,m+2}} \ldots v_m^{b_{m,m+2}} v_{m+2},\\
& \cdot \quad \cdot \quad \cdot \\
\pi_{n-m} &= v_1^{b_{1n}} v_2^{b_{2n}} \ldots v_m^{b_{mn}} v_n.
\end{aligned}$$

It follows that we can generally write (2.101) in the form (2.102) with $\sigma = n - m$.

By the theory of implicit functions we may expect to be able to solve equation (2.102) for one of the $\pi$'s in terms of the others. We can then reduce (2.102) to the form

$$\pi_1 = g(\pi_2, \pi_3, \ldots, \pi_\sigma). \tag{2.107}$$

In model experiments, where the principle of *similitude* is used, the quantities $\pi_2, \pi_3, \ldots, \pi_\sigma$ are taken the same for the two systems. These systems are then said to be *physically similar.* We can write (2.107) in the form

$$v_1 = v_2^{\beta_2} v_3^{\beta_3} \ldots v_n^{\beta_n} h(\pi_2, \pi_3, \ldots, \pi_\sigma) \tag{2.108}$$

when the exponent of $v_1$ in $\pi_1$ is different from zero. We write $h(\pi)$ for the function $h(\pi_2, \pi_3, \ldots, \pi_\sigma)$. Suppose that $v_1$ does not occur in $\pi_2, \pi_3, \ldots, \pi_\sigma$. To determine $h(\pi)$ for the prototype we need but take $v_2, v_3, \ldots, v_n$ for the model so that $\pi_2, \pi_3, \ldots, \pi_\sigma$ have the same values as for the prototype. We then measure $v_1$ for the model. Since the $v$'s are now known for the model, we can solve (2.108) for $h(\pi)$. The value of $h(\pi)$ is now determined for the prototype.

When $\sigma = 1$ relation (2.107) simply means that $\pi_1$ is a constant.

The Buckingham pi theorem has been especially useful in developing the sciences of heat transfer and fluid flow.

In order to be able to choose the primary and secondary quantities intelligently in the treatment of a particular problem it is necessary to have a considerable experimental and theoretical background in the field of this problem. Otherwise the analyst may use too many or too few variables in the study of this problem.

The $\pi$-terms are often called *dimensionless groups.*

**2.22. Dimensional analysis applied to a falling body.** Consider the case of a body falling in a vacuum near the earth's surface. Let $s$ be the distance the body falls in a time $t$, and $g$ the magnitude of the acceleration of gravity. Taking these as secondary quantities, the primary quantities are length $L$ and time $T$. We write a $\pi$-term as

$$\pi_1 = s^x t^y g^z.$$

This term has the dimensionality $L^x T^y (L/T^{-2})^z$. Since we wish to solve for $s$, we take $x = 1$. The exponents of $L$ and $T$ must vanish in $\pi_1$. We thus have

$$1 + z = 0,$$
$$y - 2z = 0,$$

from which $y = -2$, $z = -1$. It follows that $\pi_1$ is $st^{-2}g^{-1}$. Since there is only one independent $\pi$-term, this term is a constant. Using $k$ for the constant we have

$$s = kgt^2. \tag{2.109}$$

The constant $k$ can be determined from experiment. We could, of course, have obtained (2.109) with $k = \frac{1}{2}$ as a solution of the equation of motion of the body. *Dimensional analysis often leads to integrated forms of the equations of motion.*

Suppose that we had taken $s$, $t$, $g$, and $m$ as the secondary quantities, where $m$ is the mass of the body, and $L$, $T$, and mass $M$ as primary quantities. The $\pi$-term would then have been $st^yg^zm^u$. Since this term is dimensionless in $M$, the exponent $u$ vanishes. The theory of the preceding paragraph now applies.

### 2.23. Dimensional analysis applied to a vibration problem.

Consider the spring-dashpot-mass system of Figure 16 and Section 2.5. We suppose that the base (support) of the system is moved sinusoidally in the direction of the vertical so that the vertical coordinate $y$ of the base is $y_0 \sin \omega t$ for constants $y_0$ and $\omega$. It is to be understood that the coordinates $x$ of the body and $y$ of the base are measured with respect to the same vertical coordinate axis. Suppose that we wish a formula for the amplitude $x_M$ of the vibration of the body. The secondary quantities are now $x_M$, $m$, $k$, $c$, $y_0$, and $\omega$. The primary quantities are mass, length, and time, denoted by $M$, $L$, and $T$ respectively. There are thus three dimensionless $\pi$'s, designated by $\pi_1$, $\pi_2$, and $\pi_3$ respectively. The dimensions of the secondary quantities are as follows:

| QUANTITY | DIMENSIONS |
|---|---|
| $x_M$ | $L$ |
| $m$ | $M$ |
| $k$ | $MT^{-2}$ |
| $c$ | $MT^{-1}$ |
| $y_0$ | $L$ |
| $\omega$ | $T^{-1}$ |

We write a $\pi$-term as

$$x_M{}^\alpha m^\beta k^\gamma c^\delta y_0{}^\epsilon \omega^\lambda.$$

For this term to be dimensionless the exponents must satisfy the equations

$$\begin{aligned} \alpha \qquad\qquad\quad + \epsilon \qquad &= 0, \\ \beta + \gamma + \delta \qquad\qquad &= 0, \\ - 2\gamma - \delta \qquad - \lambda &= 0. \end{aligned} \tag{2.110}$$

We can choose the values of $\alpha$, $\beta$, and $\lambda$ arbitrarily, and solve for the remaining quantities. We select $\alpha$, $\beta$, and $\lambda$ as in the table

| | $\alpha$ | $\beta$ | $\lambda$ |
|---|---|---|---|
| $\pi_1$ | 1 | 0 | 0 |
| $\pi_2$ | 0 | 1 | 0 |
| $\pi_3$ | 0 | 0 | 1 |

Solution of (2.110) gives

$$\pi_1 = \frac{x_M}{y_0}, \qquad \pi_2 = \frac{mk}{c^2}, \qquad \pi_3 = \frac{c\omega}{k}.$$

By the Buckingham pi theorem we have

$$x_M = y_0 f\left(\frac{mk}{c^2}, \frac{c\omega}{k}\right) \tag{2.111}$$

for a function $f$. The function $f$ may be determined from experiment. Equation (2.111) can also be obtained by solving the equation of motion of the body.

## REFERENCES

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3. Fred B. Seely and Newton E. Ensign, *Analytical Mechanics for Engineers*, Wiley, New York, 1941.
4. P. W. Bridgman, Dimensional Analysis, Yale, New Haven, 1931.

# *Problems for Part I*

In each of the following problems the reader is expected to set up a mathematical equivalent to the physical system described. He is not expected to solve the equations so obtained.

1. One end of a spring with scale 100 lb per in is fixed, and the other attached to a body weighing 100 lb. The body is displaced 1 inch from equilibrium, with the spring distended, and then released.
2. One end of a spring is fixed and the other is attached to a dashpot plunger. The scale of the spring is 20 lb per in and the damping coefficient for the dashpot is 10 lb sec per in. Neglect the mass in the system. The spring is stretched 2 inches and then released.
3. A sphere of homogeneous material is dropped into a large body of water.
4. One end of a beam is fixed at a wall and the other end is free. A heavy body rests on the free end. This body is given a downward velocity. We are interested in the motion of this body.
5. A body is suspended from a ceiling by a spring, and a second body is attached to the first body by a spring. The system is disturbed.
6. A body is attached to a spring, which in turn is attached to a ceiling. The body is given a downward velocity so that it strikes a base of the same material head on.
7. A railroad car runs against a bumper at the end of the track.
8. A spring is attached to a wall; the other end is connected to a body. The whole system is submerged in oil. Motion of the body results from an arbitrary disturbance.
9. A marine engine is suddenly disconnected from the propeller drive shaft. The shaft is rotating. We are interested in the motion of the propeller.
10. A heavy object is dropped from an airplane in motion. We are interested in the motion of the object.
11. The crankshaft of a reciprocating engine vibrates.
12. A boat is propelled through the water. We are interested in the equation of motion of the boat.
13. A diesel with fixed throttle has a load which varies sinusoidally with time. We are interested in the speed of the engine.
14. Rotating machinery is suddenly uncoupled from the engine driving it. We are interested in the speed of the machinery.
15. A body attached to a shaft is rotating about the axis of the shaft, the other end of the shaft being fixed. Use the energy method to analyze the motion of the body.

16. A driver suddenly applies the brakes to an automobile. We are interested in the motion of the automobile.
17. A pilot suddenly moves the elevators of a plane where the plane travels "in a straight line" before the movement. Both elevators are moved in exactly the same manner. Analyze the effect on the plane.
18. A band saw driven by a diesel engine is lowered into a block of wood. We are interested in the speed of the engine.
19. A balance scale is equipped with a dashpot to make the scale come to rest in a better manner. A weight is suddenly set on the scales. Analyze the motion of the scale.
20. A continuous belt is placed over a pulley whose axle is horizontal. The axle is attached to vertical springs at its ends, which in turn are suspended from a rigid ceiling. A weight is attached to a pulley at the other end of the belt. The system is disturbed. We are interested in the motion of the weight.
21. A constant force is applied to a body attached to a base by a spring and also by a dashpot.
22. An excavating shovel is driven by a diesel engine. The shovel has a load of dirt. The throttle on the engine is moved with constant velocity toward the increased fuel position while the dirt is being raised by the shovel. What happens to the speed of the engine?
23. Two bodies attached by a stretched spring are free to slide on a dry horizontal surface. The bodies are held apart for some time and then suddenly released. We are interested in the motion.
24. An engine shaft terminates in a gear that drives a second gear. This second gear is at an end of a shaft, the other end of which is attached to the rotor of a generator. The gearing is such that the shafts run at different speeds. The engine throttle is instantly shut down. It is assumed that the generator is not coupled with others supplying the same line. We are interested in the speed of the generator.
25. A body rests on a flat circular disk. The surface of the disk is dry. The disk rotates, starting from rest, at constant acceleration. We are interested in the motion of the body.
26. The pressure $p$ in an oil-filled cylinder is varied directly as the square of the time. A piston in the cylinder on which this pressure acts moves under the influence of this pressure. The piston is connected to a piston rod which in turn is connected to bodies subject to inertia only (not subject to spring and damping forces). What happens to the piston?
27. A wheel with spokes is rotating freely on an axis. A wooden lath with one end fixed against a wall and held parallel to the axis of the wheel passes a little through the wheel so that the lath flips from one spoke to the other. Analyze the system.
28. A laboratory room is supported at four corners on rubber blocks to minimize the effect of external vibrations on the room. Passage of a nearby train vibrates the base on which the rubber blocks rest. What happens to the room?
29. A cylindrical flywheel with a chunk missing is rotating at constant angular velocity about a horizontal axle that is supported on two vertical springs, the bottom ends of the springs being fixed on a foundation. The axle is free to move vertically only. We are interested in the motion of the axle.

30. A body swings in a circle on the end of an inelastic string. We are interested in the centrifugal force.
31. The speed of a diesel engine is governed by ordinary flyballs whose outward movement is directly transmitted to the throttle. We wish the equations relating the motion of the flyballs, throttle, and the speed of the engine Figure 33 illustrates a modern diesel governor
32. The rotor of a ship gyroscope is spinning with constant angular velocity. We are interested in the gyroscopic effect.

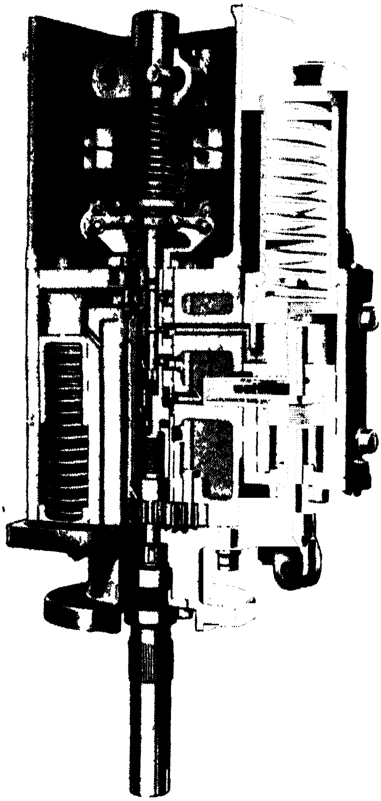

**Fig. 33. Cutaway of a diesel governor. Courtesy of the Woodward Governor Company.**

33. By dimensional analysis find a formula for the time of swing of a simple pendulum.
34. A sinusoidal force is applied to the body of Figure 16. By dimensional analysis find the amplitude of the oscillations of the body.
35. A body falls freely in a vacuum near the earth. By dimensional analysis find the speed of the body.

# PART II

# ELECTRICITY AND MAGNETISM

# CHAPTER

# 3

# *Fundamentals of electricity and magnetism*

**3.1. Inverse square law.** Part II of this book is concerned with setting up electrical and magnetic problems in mathematical form.

Michell and Coulomb verified experimentally that the inverse square law

$$|\boldsymbol{F}| = K\frac{m_1m_2}{r_{12}^2} \tag{1.8}$$

holds when the masses $m_1$ and $m_2$ are replaced by the strengths of magnetic poles. These poles are a distance $r_{12}$ apart and the force $\boldsymbol{F}$ is again directed along a line joining these poles. In (1.8) the quantity $K$ is a constant and $|\boldsymbol{F}|$ is the magnitude of the force $\boldsymbol{F}$.

The magnetic property is basically recognized through the pointing of one end of a suspended needle of lodestone or magnetite toward north. Such an end is called a *north* or *N pole*, whereas the other is a *south* or *S* pole. The strengths of two poles are said to be equal if they experience the same forces when brought into identical positions relative to a third pole. If these poles are not of the same strength we can assign numerical values to their strengths in proportion to the magnitudes of the forces they experience when brought into identical positions relative to the third pole. The mathematical theory is much simplified by using poles at points, that is, "point poles."

Although for gravitation the quantity $K$ does not depend on the medium containing the bodies with masses $m_1$ and $m_2$, the quantity $K$ does depend on the medium when $m_1$ and $m_2$ are strengths of magnetic poles.

A pole of unit strength is called a *unit pole*. It is understood that a *magnetic field* exists at the points of a region $R$ if a pole placed at any point of $R$ experiences a magnetic force acting on it. As will be seen later this force need not be due to magnetic poles but may, for example, be due to the current in a conductor.

By definition the *strength* at a point $P$ due to a magnetic field is the force $\boldsymbol{H}$ per unit pole which acts on a discrete $N$ pole placed at $P$, assuming that the pole does not affect the field. A field is given when its strength at each point is known. The strength $H$ is commonly termed the *magnetic intensity* at $P$. At a point in a field the force acting on a pole of strength $m$ is $Hm$. Here $m$ is taken as positive for an $N$ pole and negative for an $S$ pole. Let $H^*$ be the sense and magnitude of the magnetic intensity $H$ at a point $P$ a distance $r$ from a pole of strength $m$, where $H$ is the intensity at $P$ due to $m$. By "sense" we mean that if we replace $m$ by $-m$ the direction of $H$ is reversed. Now by the inverse square law

$$H^* = \frac{m}{\mu_0 r^2} \tag{3.1}$$

for a vacuum, where $\mu_0$ is a constant that depends on the system of units employed.

In actual practice a magnetic field is usually determined from measurements made with a *fluxmeter* attached to a coil in which, if the field is constant in time, an electromotive force is induced by the movement of the coil through the field. If the field is varying in time the coil is usually held stationary.

Although the inverse square law for magnetism is not ordinarily used directly in practical engineering computations it is valuable for the understanding of the subject, in particular for such concepts as magnetic potential and magnetic moment.

When certain bodies are rubbed, such as glass or amber, the presence of electric charges is recognized by the mutual attraction or repulsion between these bodies. Charges on bodies can be measured by comparing the forces between these bodies and a standard body, the measurement being made in a short period of time during which the charges on these bodies are relatively constant, that is, the charges do not leak off to the surrounding medium. Coulomb verified with pith balls that the inverse square law (1.8) is valid for electric point charges, where $m_1$ and $m_2$ are replaced by the numerical values $q_1$ and $q_2$ of these charges. There are two kinds of charges, positive and negative. Like charges repell and opposite attract each other. The force $F$ of the inverse square law (1.8) acts along the line joining the charges. Numerical values are assigned to charges just as they are assigned to the strengths of magnetic poles. The constant $K$ in (1.8) for electric charges depends on the medium in which these charges are placed.

It has been found that an *electron* carries a definite unit of negative charge, whereas *positrons* and *protons* carry the same magnitude of positive charge. The electric charge carried by a body is the sum of the charges carried by the particles in the body.

As was the case for magnetism and magnetic fields, it is often convenient to use field theory to treat electric effects. The *electric intensity* $E$ at a point $P$ due to an electric field is the force $E$ per unit charge which acts on a positive charge at $P$, assuming that the charge at $P$ does not affect the field. At a point in an electric field the force acting on a charge $q$ is $\boldsymbol{E}q$. Here $q$ is taken to be positive for a positive charge, and negative for a negative charge. Let $E^*$ be the sense and magnitude of the electric intensity $\boldsymbol{E}$ at a point $P$ a distance $r$ from the point $Q$ at which a charge $q$ is located, it being understood that $\boldsymbol{E}$ is due only to the presence of $q$. For a vacuum the inverse square law (1.8) yields

$$E^* = \frac{q}{\epsilon_0 r^2}, \tag{3.2}$$

where $\epsilon_0$ is a constant that depends on the system of units employed.

A medium which can give rise to an electric field due to the presence of its charges is termed a *dielectric*. In engineering practice the term *dielectric* is generally understood to mean a medium which is a poor conductor of electricity compared with others termed *conductors*. When an electric field is present in a dielectric, due to a shift of the positions occupied by the electrons in the atoms, the dielectric may be said to have become *charged* or *polarized*. The substitution of a dielectric for a vacuum results in a change in the intensity $\boldsymbol{E}$. Accordingly, it is necessary to replace (3.2) by the more general equation

$$E^* = \frac{q}{\epsilon_0 k r^2} \tag{3.3}$$

where $k$ is called the *dielectric constant* or *relative permittivity* of the medium. In *homogeneous isotropic media, where the same condition prevails at each point of the medium and for each direction through each point*, the quantity $k$ is a constant. In engineering computations $k$ may be taken to be 1 for air. For normal conditions the difference between $k$ for air and $k$ for a vacuum is less than $\frac{6}{100}$ of 1 per cent.

If in the case of magnetism the vacuum is replaced by a medium such as iron, equation (3.1) becomes

$$H^* = \frac{m}{\mu_0 \mu r^2}, \tag{3.4}$$

where $\mu$ is termed the *relative permeability* of the medium. In homogeneous isotropic unmagnetized media $\mu$ is a constant. The quantity $\mu$ is practically the same for air as for free space. In the case of magnetic materials, like iron and nickel, the quantity $\mu$ depends very considerably on the past history of the material due to *hysteresis*, and on other factors such as the field intensity.

Since we shall be concerned with relative permeabilities only, we shall refer to $\mu$ as the *permeability* of the medium, as is often done in engineering.

The dielectric constant $k$ ranges in ordinary applications from 1 for air to 80 for distilled water. In some less commonly encountered materials

$k$ may have much higher values. For a given substance under fixed physical conditions, such as fixed temperature and pressure, the constant $k$ depends on the frequency when the substance is subjected to an alternating field. The permeability $\mu$ of materials in ordinary applications ranges from 1 for air to as much as 275,000 for magnetic iron purified in hydrogen.

In what follows we shall generally take our units so that $\epsilon_0 = \mu_0 = 1$.

**3.2. Potential.** The study of fields is much simplified by the use of a potential function. Let $x$, $y$, and $z$ be Cartesian coordinates. Let $F_x$, $F_y$, and $F_z$ represent the components of the field intensity $F$ at a point $P$, where the field is electric or magnetic. By definition the *potential* for this field is a function $f(x, y, z)$ for which

$$F_x = -\frac{\partial f}{\partial x}, \quad F_y = -\frac{\partial f}{\partial y}, \quad F_z = -\frac{\partial f}{\partial z}. \tag{3.5}$$

To be the potential of the field the relations (3.5) must be valid at all points of the region of the field.

Relations (3.5) differ from the corresponding ones for the gravitational field in that we have a minus sign on the right.

A force field is *conservative* if a potential function exists for this field. We shall show that *a field for which the inverse square law holds is a conservative one.* This result greatly simplifies the study of gravitation, magnetism, and electricity. In the following theorem *charge* may be mass, electric, or magnetic charge. We restrict ourselves to *finite fields* for which the magnitude of the field intensity is everywhere finite in the region of space studied.

***Theorem* 3.1.** *A finite field for which the inverse square law holds is conservative, whence such gravitational, magnetic, and electric fields in space free of charges are conservative.*

We consider the field generated by a point charge $m$ at the origin of an $x$, $y$, $z$-Cartesian coordinate system. By the inverse square law the field intensity $F$ at the point $P(x, y, z)$ of Figure 34 has the magnitude $|F|$, where

$$|F| = \frac{Km}{x^2 + y^2 + z^2}, \tag{3.6}$$

provided $m$ is positive. We introduce the function $f$ where

$$f = \frac{Km}{\sqrt{x^2 + y^2 + z^2}}. \tag{3.7}$$

Differentiation of $f$ and the computation of the components $F_x$, $F_y$, and $F_z$ of $F$ show that (3.5) is satisfied, except that the minus signs do not occur when $m$ is mass. It is readily seen that (3.5) is also true when $m$ is negative.

The potential function $f$ is defined up to a constant. This constant can be chosen arbitrarily. Generally, it is convenient to choose this constant so that the function $f$ goes to zero as $(|x| + |y| + |z|)$ increases beyond all bounds.

The treatment above for a field due to a point charge at the origin $O$ can be extended by usual methods to several such charges, and to a continuous distribution of charge. As remarked above, this *charge* may be mass, magnetic strength, or electric charge. The reader should verify that there is always a potential function describing the field for continuous charge distributions over closed portions of space or surfaces of finite area. To obtain this potential at a point $P$ in a charged body one should cut out from the body a region containing $P$, obtain at $P$ the potential $V$ of the rest of the body, and take the potential at $P$ in the given body to be the limit of $V$ as the region shrinks to $P$.

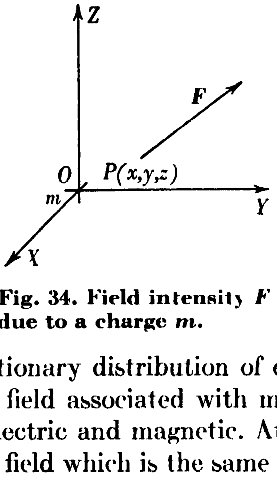

**Fig. 34. Field intensity $F$ due to a charge $m$.**

The field due to a stationary distribution of electric charges is called an *electrostatic field*. The field associated with moving charges is *electromagnetic*, that is, both electric and magnetic. At any instant $t$ this field has a component electric field which is the same as if the charges were at rest at the positions occupied at the instant $t$. The entire field is obtained by superimposing on this electric field the field due to movement of the charges.

If we take the line integral (see Section 1.9)

$$\int_A^B |F| \cos\theta ds \tag{3.8}$$

from a point $A$ to a point $B$ in a conservative force field with intensity $F$, regardless of the path of integration from $A$ to $B$, we have

$$-\int_A^B |F| \cos\theta ds = -\int_A^B (F_x dx + F_y dy + F_z dz) = \int_A^B df = (f)_A^B \tag{3.9}$$

for the potential function $f(x, y, z)$. The integral in (3.8) is the work done in moving a unit point charge from $A$ to $B$. It is clear that this work is independent of the path along which this charge is moved. Since

$$\int_B^A |F| \cos \theta ds + \int_A^B |F| \cos \theta ds = 0, \tag{3.10}$$

the work done in moving the charge around a closed path is zero, that is, no energy has been dissipated in this process, whence we have the expression *conservative field.*

*The work per unit charge done in moving a charge from one point A to another point B in a conservative field is equal to the difference of potential* $(f_B - f_A)$ *between these points.*

**3.3. A connection between electricity and magnetism.** We let $q$ be a positive electric charge moving with constant velocity $v$ as shown in Figure 35, and let $P$ be a point a distance $r$ from $q$. We have:

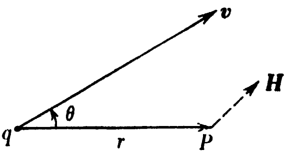

**Fig. 35. Magnetic field intensity *H* due to a moving charge *q*.**

*Law* **3.1.** *The positive charge q sets up a magnetic field with intensity H where*

$$|H| = \frac{q|v| \sin \theta}{r^2}, \tag{3.11}$$

*for an angle θ between the line through q and P and the line of the velocity v; the direction of H in Figure 35 is perpendicular to the plane of the paper and away from the reader.*

Let $r$ be the vector from $q$ to $P$. We can replace (3.11) by the vector form

$$H = \frac{qv \times r}{|r|^3}.$$

If $q$ in (3.11) is replaced by a negative charge, formula (3.11) applies with $q$ replaced by $|q|$, but the *direction of H is reversed.* The angle $\theta$ is by convention taken between 0° and 180°.

Law 3.1 has been credited to Biot and Savart, and was stated by them for electric currents.

A drift of electric charges in a given direction is understood to be a *current* in this direction. If a current flows through a body, the body is called a *conductor* of this current. A current through a wire is to be recognized by the presence of a magnetic field around the wire, or the heating of the wire under certain conditions (that is, when the heating is due to electric effects and not to chemical action within the conductor, an external heat source, or some other cause, which might be mechanical as when the bending of a wire produces friction).

A current through a wire is due to the drift of electrons, which are extremely light particles carrying negative charges. This is proved by accelerating a wire rapidly. With very sensitive measuring instruments it is found that a current is thus generated in the wire. Calculations indicate that this current is due to the inertia of electrons in the wire.

It is customary in electrical engineering to take the direction of positive current in a wire as opposite to the drift of electrons, that is, opposite to the true current. We follow this tradition here.

A drift of electrons implies *free* electrons not fixed in atoms.

A current may be due to flow of positively charged bodies, such as positive ions traveling in a solution, called an *electrolyte*, from one electrode to another. We shall, however, treat only currents due to electrons, the theory for currents of positively charged bodies being an immediate consequence with appropriate changes of sign.

Although charges occur in discrete quantities, and there is no such thing as a continuous charge distribution in modern science, the substitution of a continuous charge distribution for a set of discrete charges yields satisfactory mathematical results unless we are concerned with atomic distances. We may therefore use the limit methods of the calculus for ordinary phenomena.

When electrons drift in a conductor there are many collisions between them and the atoms in the conductor. The resultant agitation of the atoms exhibits itself as heat.

If a surface does not bound a region of space, one of the sides of the surface may arbitrarily be taken as the outward side. By definition the *current* $i$ through a surface at a given instant is the rate at which charge is passing outward across the surface, that is,

$$i = \frac{dq}{dt} \tag{3.12}$$

for the charge $q$ and time $t$. We can cross the surface at a point in two normal directions, and the sign of $i$ will vary accordingly.

If a current $i$ flows along a conductor for which the maximum chord of a cross section $S$ is negligible compared to the distance from $S$ to a given point $P$ at which the magnetic field intensity due to $i$ is to be obtained, the Biot-Savart theorem can be stated in its usual form to be given below. In practical cases the word "negligible" usually applies where the ratio of maximum chord length to the distance $SP$ is $\frac{1}{100}$ or less. We write the current $i$ at the cross section $S$ as in equation (3.12), and let $l$ be a coordinate of distance along the conductor. We let $\Delta l$ be the *average* distance the charges assumed to be at or near $S$ at the instant $t$ travel during an interval of time $\Delta t$ after the instant $t$. We may chose $l$ so that $\Delta l$ is positive. The charges at $S$ at the time $t$ are assumed to be at the section $S^*$ at the instant $(t + \Delta t)$ as shown in Figure 36. We let $\Delta q$ be the total charge in the region $R$ of the conductor between the surfaces $S$ and $S^*$.

We shall assume that the current $i$ is constant or slowly varying. What we can consider as slowly varying is to be understood from Maxwell's equations to be given later. For many practical considerations we may suppose that the frequencies of up to 440 cycles per second used in power engineering are slowly varying, as is often also true for the 0 to 5,000 cycles per second commonly used in voice transmission. We shall also assume that the points at which we wish to determine the magnetic field intensity are short distances from the source of disturbance of the field if the field is not constant. Experiments verify that magnetic, and also electric, field disturbances, usually called *waves*, propagate in a vacuum with the speed $c$ of light, $c$ being about 186,300 miles per second. For short distances, such as a foot or a mile, and considerations involving long enough time intervals, such as at least 1/10,000th of a second, we may suppose, often with small error, that waves propagate instantaneously from the source to the points studied, so that the field always corresponds to the source.

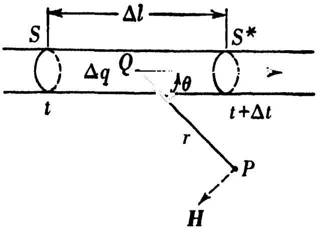

**Fig. 36. Magnetic field due to charges traveling along a conductor at a constant average velocity.**

The charges making up $\Delta q$ have an *average speed* $\eta(\Delta l/\Delta t)$, where the limit of $\eta$ as $\Delta t \to 0$ is assumed to be 1. We let $P$ be a given point outside $R$, and let $r$ be the distance from a point $Q$ in the region $R$ to $P$. Let $\theta$ $(0 \leqq \theta < 180°)$ be an angle between the line joining $P$ and the point $Q$ and a line at $Q$ specifying the direction of the conductor at $Q$. By Law 3.1 the magnetic field intensity $\Delta \boldsymbol{H}$ at $P$ due to $\Delta q$ is such that

$$|\Delta \boldsymbol{H}| = |\Delta q|\eta \frac{\Delta l}{\Delta t}\frac{\sin\theta}{r^2} \tag{3.13}$$

with the direction prescribed in Law 3.1, and proper choices of $Q$ (hence $\theta$) and $\eta$.

Division by $\Delta l$ and the taking of the limit at $\Delta t$ goes to zero yields

$$\left|\frac{dH}{dl}\right| = \left|\frac{dq}{dt}\right| \frac{\sin\theta}{r^2}, \tag{3.14}$$

assuming that the derivatives exist. Multiplying both sides of (3.14) by $dl$ and replacing $dq/dt$ by $i$, we have (3.15) to follow.

***Theorem* 3.2. Biot-Savart Theorem.** *If a constant or slowly varying current $i$ flows through a conductor of negligible cross section, the magnitude of the differential of the magnetic intensity $\mathbf{H}$ at a point $P$, where $\mathbf{H}$ is a function of arc length $l$ along the conductor, is given for a point $Q$ on the conductor by*

$$|d\mathbf{H}| = |i| \cdot |dl| \frac{\sin\theta}{r^2}, \tag{3.15}$$

*where $r$ is the distance from the point $Q$ to the point $P$, and $\theta$ is an angle ($0 \leq \theta < 180°$) between the direction of the current at $Q$ and the line $QP$. The vector $d\mathbf{H}$ is perpendicular to the plane of the tangent line to the conductor at $Q$ and the line $QP$. If the conductor is grasped by the right hand, with the thumb extended in the direction of current, the direction of the fingers indicates the direction of $\mathbf{H}$.*

From the mathematical point of view it is convenient to replace a conductor of *negligible* cross section by a curve, carrying the same current, in fact, the word *negligible* means that the conductor can be replaced by a curve with the same practical results.

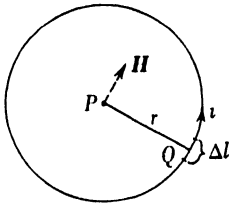

**Fig. 37. Magnetic field intensity at the center of a wire loop.**

There is a vector form

$$d\mathbf{H} = \frac{i d\mathbf{l} \times \mathbf{r}}{|\mathbf{r}|^3}$$

of the above theorem.

One application of the Biot-Savart theorem is to the determination of the field at the center $P$ of a circular wire of small cross section. We consider a loop of radius $r$ as shown in Figure 37. We let $l$ be arc length measured from some point on the loop, and note that $\theta = 90°$ for each point on the loop. For $i$ measured positively we thus have

$$|\,H\,| = \int_{l=0}^{l=2\pi r} |\,dH\,| = \frac{i}{r^2}\int_0^{2\pi r} dl = \frac{2\pi i}{r}. \tag{3.16}$$

For an ordinary short coil with $n$ turns, the intensity is $nH$.

A commonly used unit of charge is the *coulomb*. One electron carries a charge of approximately $1.59 \cdot 10^{-19}$ coulombs. A *current of one ampere* means a rate of flow of charge of one coulomb per second, that is, a flow of about $6.3 \cdot 10^{18}$ electrons per second. Although individual electrons in a conductor may be traveling as fast as $10^7$ inches per second, the velocity of drift in a copper wire 0.04 inch in diameter carrying one ampere will be about 0.004 inch per second.

**3.4. Flux and lines of force.** A field of vectors is not a particularly convenient concept. It is simpler in many respects to think of *lines of force*, namely curves so drawn that at each point on the curve the line of the field intensity coincides with the tangent line to the curve at the point. Such lines are also called *lines of flux*. In a magnetic field such lines can sometimes be visibly formed by iron filings sprinkled in the field.

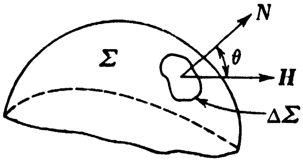

Fig. 38. Magnetic flux across a surface Σ.

The field is known when the lines of force are known, as well as the sense of the force along each tangent line, and the numerical magnitude of the force. The sense is usually known from simple physical considerations. The magnitude in the case of a magnetic field is known from the permeability $\mu$, and the flux defined as follows.

We let $\Sigma$ be a surface in the field with a tangent plane at each point, and let $\Delta\Sigma$ be an element of area on $\Sigma$ as shown in Figure 38. We let $H$ be the magnetic field intensity at a point on the element, and let $\theta$ be an angle between $H$ and the *outward* normal $N$ to the surface $\Sigma$ at this point. The *flux* $\psi$ through the surface $\Sigma$ is given by

$$\psi = \iint_{\Sigma} \mu |\,H\,| \cos \theta d\Sigma, \tag{3.17}$$

where $|\,H\,|$ is the magnitude of the vector $H$. Vectorially, we have

$$\psi = \iint \mu H \cdot d\Sigma.$$

If $\mu$ and $\boldsymbol{H}$ are constant at all points of the surface $\Sigma$, and $\Sigma$ is a part of a plane perpendicular to the direction of $\boldsymbol{H}$, the relation (3.17) becomes

$$\psi = \mu|\boldsymbol{H}|A, \tag{3.18}$$

where $A$ denotes the area of $\Sigma$, and $\theta$ has been taken equal to zero. For unit area

$$\psi = \mu|\boldsymbol{H}|. \tag{3.19}$$

The flux $\psi$ is thus a measure of $\mu\boldsymbol{H}$.

It may happen that $\mu$ and $\boldsymbol{H}$ vary from point to point. Assume that $\mu$ and $\boldsymbol{H}$ vary continuously (as functions of position) over the surface $\Sigma$, and let $\Sigma$ include its boundary if it has one. By the first theorem of the mean of the integral calculus we can write $\psi$ of (3.17) as

$$\psi = A[\mu|\boldsymbol{H}|\cos\theta]_P, \tag{3.20}$$

where $[\mu|\boldsymbol{H}|\cos\theta]_P$ is the value of $\mu|\boldsymbol{H}|\cos\theta$ at a point $P$ on $\Sigma$, and $A$ is the area of $\Sigma$. The surface $\Sigma$ may not have a boundary, such as in the case of a sphere, or may have one like a hemisphere. Relation (3.20) implies that $\psi = 0$ when $A = 0$, provided the coefficient of $A$ in (3.20) is different from zero. A mere change of notation yields

$$\Delta\psi = [\mu|\boldsymbol{H}|\cos\theta]_{P_1}\Delta\Sigma \tag{3.21}$$

for a point $P_1$ on a given element of surface $\Delta\Sigma$ on the surface $\Sigma$. Let $\delta$ be the maximum length of the chords joining points of this element. Divide both sides of (3.21) by $\Delta\Sigma$, and take the limit as $\delta$ goes to zero, and the element shrinks to a point $P$. The point $P_1$ then approaches $P$ and relation (3.21) yields

$$\frac{d\psi}{d\Sigma} = \mu|\boldsymbol{H}|\cos\theta \tag{3.22}$$

valid at $P$. The area $\Delta\Sigma$ goes to zero with the maximum chord length. If the normal at $P$ is in the direction of the vector $\boldsymbol{H}$ at $P$, relation (3.22) becomes

$$\frac{d\psi}{d\Sigma} = \mu|\boldsymbol{H}|. \tag{3.23}$$

We have shown that *the rate of change of flux with respect to area, on a surface normal to the magnetic field intensity* $\boldsymbol{H}$ *at a point* $P$, *is* $\mu|\boldsymbol{H}|$, the angle $\theta$ being taken equal zero. When physicists and engineers speak of *flux in lines per unit of area,* they really mean the derivative $d\psi/d\Sigma$.

In a field where $\mu$ and $\boldsymbol{H}$ are constant, $d\psi/d\Sigma$ is the flux $\psi$ per unit area, the area being measured on a plane as for the formula (3.19), and with the correct sense of the outward normal.

We introduce the *magnetic induction* $\boldsymbol{B}$, *where*

$$B = \mu \boldsymbol{H}. \tag{3.21}$$

The magnitude of $\boldsymbol{B}$ is $d\psi/d\Sigma$ given in (3.23). It is customary in engineering circles to use the symbol $B$ for a function representing the magnitude and sense of $\boldsymbol{B}$, and to speak of the magnitude of $\boldsymbol{B}$ as *flux density*, measuring $B$ numerically in lines per square inch. *The magnetic induction* $\boldsymbol{B}$ *depends on the point at which* $B$ *is evaluated, and the direction of* $\boldsymbol{B}$ *at this point.*

A force of $10^6$ *dynes* is the same, for practical purposes, as 2.248 pounds. Using $\mu = 1$ for empty space, if $\boldsymbol{H}$ is in dynes per unit pole, the flux density is by definition in lines per square centimeter. One inch is approximately 2.54 centimeters.

In engineering the magnetic field intensity $\boldsymbol{H}$ is often called the *magnetizing force*. The magnitude and sense of $\boldsymbol{H}$ are represented by $H$. The magnitude of $\boldsymbol{H}$ is normally measured in ampere turns per inch for reasons to be explained in a later section.

The treatment of flux above applies in the same way to electric fields with

$$\psi = \iint_{\Sigma} k|\boldsymbol{E}| \cos \theta d\Sigma \tag{3.25}$$

for the *electric flux* $\psi$, dielectric constant $k$, and electric field intensity $\boldsymbol{E}$. Vectorially, (3.25) can be written as

$$\psi = \iint_{\Sigma} k\boldsymbol{E} \cdot d\boldsymbol{\Sigma}.$$

Corresponding to $\boldsymbol{B}$ there is an *electric induction* $\boldsymbol{D}$ for which

$$\boldsymbol{D} \doteq k\boldsymbol{E} \tag{3.26}$$

The vector $\boldsymbol{D}$ is also called the *electric displacement*, and *electric flux density*.

Lines of force are sometimes replaced by *tubes of force*. These are merely geometric concepts to visualize the meaning of (3.23) and the analogous relation

$$\frac{d\psi}{d\Sigma} = k|\boldsymbol{E}|.$$

Although the dielectric constant $k$ is in common use in numerical computations, this is not as much true of $D$ and $E$. The concepts of $\boldsymbol{D}$ and $\boldsymbol{E}$ are invaluable for understanding the theory of dielectrics. The quantities $B$ and $H$ for magnetism are in constant use.

Although the dielectric constant $k$ is, for practical purposes, constant for a given homogeneous material under given physical conditions of temperature, pressure, and rate of change $dE/dt$ of $\boldsymbol{E}$ with respect to time, this is not true for the permeability $\mu$. Even for a steady state, the value of $B$ for a given $\boldsymbol{H}$ is not unique. When we go from an intensity $\boldsymbol{H}$ through a series of intensities and then back to $\boldsymbol{H}$, the final value of $B$ may be different from its initial value. This phenomenon is termed *hysteresis.* Such a path for iron is given in Figure 39 where $H = 0$ at the origin $O$ and increases to $H_{\max}$ at $A$ and drops back to zero at $C$. The value of $B$ at $C$ is different from zero. Hysteresis results from the fact that considerable energy is lost in this process. This energy cannot be reclaimed by restoring the original $\boldsymbol{H}$.

Magnetic properties of materials are usually described by characteristic curves as in Figure 40. In this figure $B$ is in kilogausses, one kilogauss being 1,000 gausses, and one *gauss* being approximately 6.45 lines per square inch. The intensity $H$ is in oersteds. One ampere turn per inch is approximately 0.495 *oersteds.* Unfortunately, many other terms for $B$ and $H$ occur in the literature. The reader should become familiar with conversion tables.

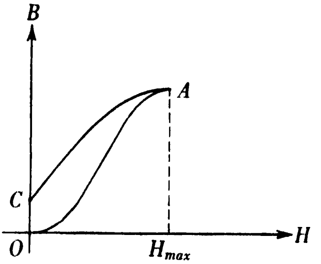

**Fig. 39. Hysteresis curve.**

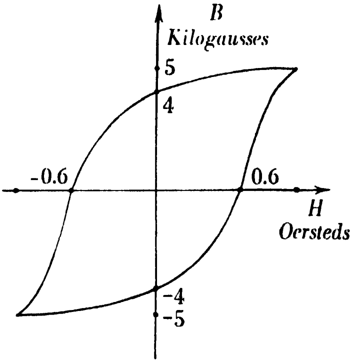

**Fig. 40. Characteristic curve of armco iron (hysteresis loop).**

It can be shown that the area of the *hysteresis loop* formed by going from a pair of values of $B$ and $H$ back to this pair, as in Figure 40 for example, is proportional to the *hysteresis loss* of energy in going around this loop.

If $B$ is the flux density in a vacuum for a magnetic field intensity $\boldsymbol{H}$, the same flux density $B$ is obtained in a medium with permeability $\mu$ by using a field intensity $\boldsymbol{H}/\mu$; that is, with $\mu > 1$ much less magnetizing force is needed to produce a given flux in the medium than is required for empty space. It is because of this fact that magnetic materials are so desirable in many applications, cutting down the size of the equipment otherwise needed. Similar remarks hold for dielectrics.

There are materials such as permalloy and hypernik for which the hysteresis loops have very small area, so that in many practical examples a hysteresis loop may be considered to be an arc of a curve.

The normal magnetizing force of the earth is about 1 or 2 lines per square inch in contrast to the 60,000 lines per square inch often used in direct current motors and generators. The variation in the earth's magnetic field is so slow that, unless we are interested in weak magnetic fields over long periods of time, we may treat this field as constant, so that in equations of force the earth's magnetic field plays a role like gravity in Section 2.1.

From the definition of flux above, a number of valuable theorems can be proved. Among them is the following.

***Theorem* 3.3. Gauss's theorem.** *The electric flux across a closed surface $\Sigma$ is $4\pi$ times the charge $q$ bounded by the surface, that is,*

$$\iint_{\Sigma} kE \cdot d\Sigma = 4\pi q$$

We consider a positive charge $q$ at the center of a sphere $\Sigma$ of radius $r$ as in Figure 41. For each point on the sphere the intensity $E$ due to $q$ is in the direction of the normal to the sphere at this point, whence $\theta$ in (3.25) is zero. By (3.3) with $\epsilon_0 = 1$ we have

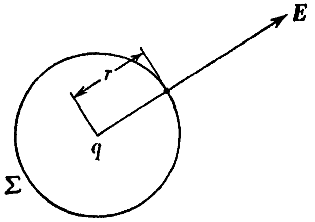

**Fig. 41. Charge $q$ at the center of a sphere.**

$$|E| = \frac{q}{kr^2} \tag{3.27}$$

at each point on the sphere. Thus (3.25) becomes

$$\psi = \frac{q}{r^2} \iint_{\Sigma} d\Sigma. \tag{3.28}$$

Since the area of the sphere is $4\pi r^2$, relation (3.28) gives

$$\psi = 4\pi q. \tag{3.29}$$

Similarly, for a negative charge $q$.

Except where otherwise noted, we shall assume in the rest of this section that the dielectric constant $k$ is the same at all points of the region considered.

Let $V$ be a region free of charges and bounded by a surface $\Sigma$ assumed to have a continuously changing normal at each point. We let $df/dn$ be the normal derivative of the potential of the field at a point on $\Sigma$. By Green's theorem of the calculus for space we have

$$\iint_{\Sigma} \frac{df}{dn}\, d\Sigma = \iiint_{V} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dV.$$

Since the potential function $f$ satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

and $df/dn$ is $|\boldsymbol{E}| \cos \theta$, the flux across $\Sigma$ vanishes.

Consider a surface $\Sigma_1$ bounding a simply connected region $V$, where $\Sigma_1$ has a continuously changing normal at each point. Suppose that the origin is an interior point of $V$, and that $V$ contains a single point charge $q$ at the origin. We let $\Sigma_2$ be a sphere of radius $r$ with the origin as center and lying within $\Sigma_1$. In the region between $\Sigma_1$ and $\Sigma_2$ Laplace's equation is satisfied. Assuming that Green's theorem applies to the region $V^*$ between $\Sigma_1$ and $\Sigma_2$ we have

$$\iint_{\Sigma_1} \frac{df}{dn}\, d\Sigma + \iint_{\Sigma_2} \frac{df}{dn}\, d\Sigma = \iiint_{V^*} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dV = 0,$$

where the integration over each surface is taken with the normal outward with respect to the region $V^*$. If the integration over the sphere $\Sigma_2$ is performed with the normal outward with respect to this sphere we have

$$\iint_{\Sigma_1} \frac{df}{dn}\, d\Sigma = \iint_{\Sigma_2} \frac{df}{dn}\, d\Sigma,$$

so that Gauss's theorem is valid for the surface $\Sigma_1$.

Suppose that we have charges $q_1, q_2, \ldots, q_n$ at points which are distances $r_1, r_2, \ldots, r_n$ from a given point $P$, respectively. The potential at $P$ due to $q_i$ is $q_i/kr_i$. By the principle of superposition, the potential at $P$ due to all of the charges is

$$\sum_{i=1}^{n} \frac{q_i}{kr_i}.$$

If we have a surface $\Sigma$ enclosing several point charges, we can enclose these charges by nonoverlapping surfaces such as spheres lying entirely within the surface $\Sigma$. We suppose that Green's theorem for space applies to the region between $\Sigma$ and the surfaces enclosing the charges. We then find that Gauss's theorem is satisfied for $\Sigma$.

Suppose that we have a continuous charge distribution over a region $V$ of space, and that we recognize the existence of this charge by an electric field in space, where the field intensity is $\boldsymbol{E}$. Let $q$ be the charge in the region $V$. We can define $q$ so as to satisfy Gauss's theorem, that is

$$\iint_{\Sigma} k|\boldsymbol{E}| \cos \theta d\Sigma = 4\pi q,$$

where we have used the notations of (3.25), and $\Sigma$ is the surface bounding $V$, it being assumed that the integral exists. Here $k$ may vary from point to point.

Let $l$, $m$, and $n$ be direction cosines of an outward normal to $\Sigma$, and let $E_x$, $E_y$, and $E_z$ designate the components of the electric intensity $\boldsymbol{E}$. By Green's theorem for space and the definition of the divergence of the vector $\boldsymbol{E}$ we have

$$\begin{aligned} k\iint_{\Sigma} \boldsymbol{E}\cdot d\boldsymbol{\Sigma} &= k\iint_{\Sigma} |\boldsymbol{E}| \cos \theta d\Sigma = k\iint_{\Sigma} (E_x l + E_y m + E_z n) d\Sigma \\ &= k\iiint_{V} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dV = k\iiint_{V} \operatorname{div} \boldsymbol{E} dV = 4\pi q. \end{aligned}$$

Suppose that div $\boldsymbol{E}$ exists and is continuous over the region $V$. By the first theorem of the mean of the integral calculus we have

$$\iiint_{V} \operatorname{div} \boldsymbol{E} dV = V_0[\operatorname{div} \boldsymbol{E}]_Q$$

for the volume $V_0$ of $V$ and the divergence of $\boldsymbol{E}$ evaluated at a properly chosen point $Q$ of $V$. By definition the charge $q$ is $\rho_0 V_0$ for an *average* value $\rho_0$ of the density $\rho$, where $\rho_0$ is some value between the maximum and minimum values taken on by $\rho$ for the region $V$. Thus

$$k[\operatorname{div} \boldsymbol{E}]_Q = 4\pi\rho_0.$$

Taking the limit as the region $V$ shrinks to a point $P$, assuming that this limit exists, we have

$$\operatorname{div} \boldsymbol{E} = \frac{4\pi\rho}{k} \tag{3.30}$$

for the divergence of $\boldsymbol{E}$ and density $\rho$ at the point $P$. For charges in empty space we have simply

$$\operatorname{div} \boldsymbol{E} = 4\pi\rho. \tag{3.31}$$

We have treated charges distributed over a region $V$ of space. We can also have surface distributions of charge.

Gauss's theorem is valid for a rather general class of surfaces.

The proof of Gauss's theorem holds for magnetic as well as electric fields. The divergence relation (3.30) is valid for any medium.

**3.5. Electromotive and magnetomotive forces.** We consider a conductor in which current is flowing from a section $A$ to a section $B$. It takes some work to bring a charge from section $A$ to $B$ because of collisions between the drifting charges, of which the current is thought of as composed, and the material of the conductor. The chord of maximum length joining points of $A$ will be assumed to be small compared with the minimum distance from points on $A$ to points on $B$, and the same is assumed to be true for section $B$. We make this assumption so that we can treat $A$ and $B$ as *equipotential sections* in what follows, that is, sections where all points are at the same potential.

The work done by a force of one dyne moving through one centimeter is one *erg*. By definition a *joule* is $10^7$ ergs. If one joule of work must be expended to move a positive charge of one coulomb between a point $P$ on $A$ and a point $Q$ on $B$, the difference of potential between $P$ and $Q$ is one *volt*.

The following law has been found to be true experimentally.

***Law* 3.2. Ohm's law.** *If $e$ is the potential difference between sections of a metal conductor (or the electrodes of an electrolytic conductor) due to the resistance of the conductor to the current $i$ flowing between these sections (or electrodes), the current $i$ is proportional to $e$, that is,*

$$e = Ri. \tag{3.32}$$

The constant $R$ in (3.32) is termed the *resistance* of the conductor. In engineering practice we may assume that Ohm's law holds in general for conductors, except that $R$ depends to a certain extent on physical conditions, such as temperature.

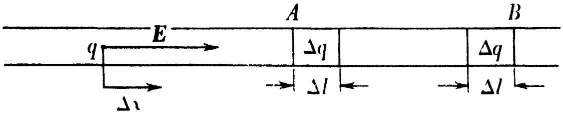

Fig. 42. Conductor.

The difference of potential $e$ between two points of a conductor is not a force, although it unfortunately is sometimes called an *electromotive force* (emf). The potential difference $e$ is also termed a *voltage*.

Let us consider a positive unit charge $q$ which moves along a conductor under the influence of an electric force intensity. When the charge $q$ is moved a straight line distance $\Delta x$ along the conductor as in Figure 42 the work done is the difference $\Delta e$ of potential between the starting point and the end point. We let $E$ designate the component of the electric force intensity in the direction of the path of $q$, it being the component that does the work. The work done in moving $q$ the distance $\Delta x$ is thus $|\, \boldsymbol{E}\, | \Delta x$ where $\boldsymbol{E}$ is evaluated at some point along the path, provided $\boldsymbol{E}$ varies continuously along the path. It follows that

$$\left|\frac{\Delta e}{\Delta x}\right| = |\, \boldsymbol{E}\, |. \tag{3.33}$$

Taking the limit as $\Delta x$ goes to zero yields (if the limit exists)

$$|\, \boldsymbol{E}\, | = \left|\frac{de}{dx}\right|. \tag{3.34}$$

Thus the magnitude of the rate of change of potential with respect to distance in a given direction yields the magnitude of the electric force intensity component in this direction. In this sense *the use of the potential is somewhat equivalent to working with a force.*

When a battery, generator, or some other source of energy maintains a difference of potential of $e$ volts across the terminals of an electric circuit we say that it maintains an electromotive force of $e$ volts.

If in (3.32) the current $i$ is in amperes and $e$ is in volts, the resistance $R$ is in *ohms*.

For materials of the same geometrical dimensions it is known that $R$ can vary over a range 1 to $4 \cdot 10^{12}$. Resistors with a resistance of $10^8$ ohms and only 1-inch long and $\frac{1}{4}$-inch wide are in commercial use. Lower resistances require even smaller elements. The wide range of $R$ makes resistors very convenient to work with. See Figure 43 for a photograph of some commercial resistors.

For magnetism there is a *magnetomotive force* $\mathfrak{F}$ where

$$\left|\frac{d\mathfrak{F}}{dx}\right| = |\, \boldsymbol{H}\, |. \tag{3.35}$$

If $\boldsymbol{H}$ is in oersteds, and $x$ in centimeters the quantity $\mathfrak{F}$ is in gilberts. One oersted is the same as one gilbert per centimeter. Engineers often prefer to measure $\mathfrak{F}$ in ampere turns. One *gilbert* is approximately 0.7958 *ampere turns*.

The *reluctance* $\mathcal{R}$ of a magnetic circuit is the capacity of a circuit to oppose the setting up of a magnetic field. By definition $\mathcal{R}$ satisfies the relation

$$\mathfrak{F} = \mathcal{R}\psi, \tag{3.36}$$

analogous to Ohm's law, where $\mathfrak{F}$ is the magnetomotive force on the circuit and $\psi$ is the flux across a section of the circuit. We note that the influence which normally magnetizes a piece of iron is external to the iron, and we may consider the iron to contain no magnetic charges. By Gauss's theorem the total flux through the closed surface formed by the part of the mag-

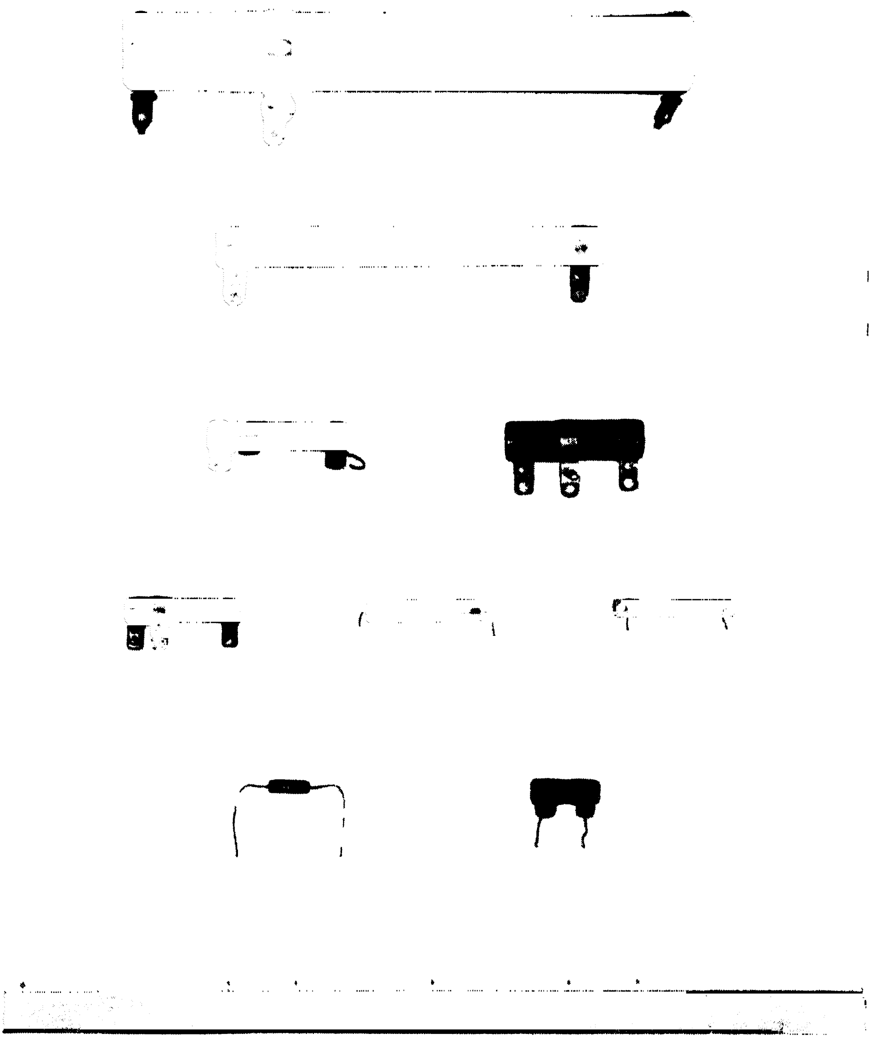

**Fig. 43. A group of industrial resistors with ratings of from 5 to 100,000 ohms. The largest has a resistance of 250 ohms, and the smallest 100,000 ohms. An inch ruler is included to show the sizes of the resistors. Courtesy of the Woodward Governor Company.**

netic circuit between sections *A* and *C* as in Figure 44 is zero. Neglecting leakage flux, that is, assuming that the flux through the lateral surface of the circuit is zero, the flux at *A* is the same as the flux at *C*. Thus, without leakage flux the flux through each cross section of the magnetic material is the same as for each other section. For example, if 100,000 lines of flux pass through *A* this is also true of *C*. The same mathematical techniques used in treating Ohm's law may thus be applied to (3.36), so that we can think of magnetic circuits in the same way as electric circuits of resistors.

If the area of section $C$ in Figure 44 is twice the area of $A$, and the flux density at $A$ is 20,000 lines per square inch, the flux density at $C$ is 10,000 lines per square inch, neglecting leakage. The problem of leakage is ordinarily handled by first leaving it out of consideration, and then making a correction for it. If a magnetic path is entirely composed of magnetic material, such as iron, the leakage is normally quite small. If the path contains an air gap, the fringing due to spreading out of the flux in the air gap must usually be accounted for.

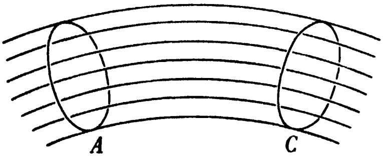

**Fig. 44. Flux through sections of a magnetic circuit.**

Suppose that the magnetic circuit is 20 inches long (length of an average path around the circuit), that the flux density is 90,000 lines per square inch, and that the magnetic material is 4 per cent silicon steel. We assume that the whole path is through this steel. A typical $B$-$H$ curve for this steel shows that $H$ must be 25 ampere turns per inch to give this flux density, whence the magnetomotive force $\mathfrak{F}$ must be 500 ampere turns. If a coil of 5,000 turns is wound around the steel (see Section 3.12), we need a current of 0.1 ampere to produce the given flux.

**3.6. Heat loss in a conductor.** We consider two surfaces (cross sections) $A$ and $B$ bounding a part of a conductor for which the current is the same at each cross section, or practically so, and for which the potential difference $e$ between the two surfaces $A$ and $B$ is a constant, the surfaces being equipotential. Consider an interval of time $\Delta t$. In this interval of time the charge $\Delta q$ crosses the surface $A$ as shown in Figure 42, the average speed of the charges being $\Delta l/\Delta t$ for a length coordinate $l$. In Figure 42 the charge $\Delta q$ crossing $A$ in the time $\Delta t$ is pictured as occupying an element of length $\Delta l$ at the end of the interval $\Delta t$, and the charge $\Delta q$ crossing the section $B$ in the time $\Delta t$ is also pictured as occupying a similar element at the right end of the conductor part being treated at the beginning of the time $\Delta t$. We are assuming that the time $\Delta t$ is taken so small that the spaces occupied by these charges do not overlap. We let $n$ and $\Delta l$ be such that the distance from $A$ to $B$ is $n\Delta l$, whence the charge $\Delta q_1$ always remaining between $A$ and $B$ during the time $\Delta t$ is $(n - 1)\Delta q$. We let $\Delta e$ be the difference in potential between the ends of a length element so that we may suppose that $e = n\Delta e$. The total work $\Delta W$ done in the time $\Delta t$ on (or by) the charges originally between $A$ and $B$ is $n\Delta q\Delta e$, or simply $e\Delta q$. Thus we have

$$\frac{\Delta W}{\Delta t} = e\,\frac{\Delta q}{\Delta t}. \tag{3.37}$$

Taking the limit as $\Delta t$ goes to zero, and writing $i$ for $dq/dt$, and $Ri$ for $e$, we obtain

$$\frac{dW}{dt} = ei = Ri^2. \tag{3.38}$$

*Power* $p$ is the derivative of work with respect to time. It follows that the power consumed by the conductor is such that

$$p = ei. \tag{3.39}$$

If $i$ is in amperes and $e$ is in volts, the power $p$ is in *watts*. A common unit of electrical power is 1,000 watts, or one *kilowatt* (kw). The power $p$ is normally evidenced as *heat*, and is due to the motion of the charge carrying bodies in the conductor.

Although energy is needed to set up a magnetic or electric field, no power is needed to maintain it as long as the field does no work. The power consumed by a resistor goes normally only into heat.

**3.7. Force on a conductor in a magnetic field.** We consider a charge $q$ moving as in Figure 45 with velocity $v$. By Law 3.1 there is at each point $P$ a magnetic force intensity $H$, satisfying (3.11), if $q$ is positive, which will be assumed to start with.

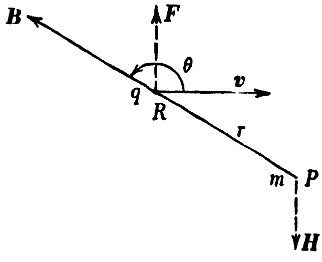

**Fig. 45. Force $F$ on a moving point charge $q$.**

We place a magnetic pole of strength $m$ at $P$ so that a force $Hm$ acts on this pole. By Newton's third law of motion a force $F$ acts on the charge $q$ in the same direction but oppositely sensed, that is

$$|F| = |H|m = \frac{mq|v|\sin\theta}{r^2} = |B||q||v|\sin\theta, \tag{3.40}$$

where by (3.4) and (3.24) the vector $B$ is the magnetic induction at the charge due to the pole $m$, and we have taken $\mu_0 = 1$. In Figure 45 $F$ is toward the reader, while $\boldsymbol{H}$ is into the paper. In what follows *we also take the constant of proportionality* $\epsilon_0$ *equal to 1.*

Let us suppose that the charge $q$ moves in a given magnetic field. At a given instant, $q$ is at a point $R$ in space for which the magnetic induction is a given vector $\boldsymbol{B}$. We can always choose $P$, and $m$ (in fact, either $m$ or $r$, or both), so that we can think of $\boldsymbol{B}$ as due to the pole $m$. We have proved the following theorem.

***Theorem* 3.4.** *If a positive charge $q$ moves with velocity $\boldsymbol{v}$ in a magnetic field with magnetic induction $\boldsymbol{B}$ (flux density) at the charge, there is a force $\boldsymbol{F}$ acting on the charge given by*

$$|\boldsymbol{F}| = |\boldsymbol{B}|q|\boldsymbol{v}| \sin\theta, \tag{3.41}$$

*where $\theta$ is an angle between the line of $\boldsymbol{v}$ and the direction of $\boldsymbol{B}$, and the direction of $\boldsymbol{F}$ is perpendicular to the plane of $\boldsymbol{B}$ and $\boldsymbol{v}$ as shown in Figure 45.*

In vector form (3.41) is

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}.$$

In Figure 45 the force $\boldsymbol{F}$ is toward the reader. For a negative charge $q$ the direction of $\boldsymbol{F}$ is reversed. Since we usually work with currents the following form of Theorem 3.4 is stated.

***Theorem* 3.5.** *If a current $i$ flows through a straight line conductor, and the conductor is placed in a magnetic field with the same magnetic induction $\boldsymbol{B}$ (flux density) at each point on the conductor perpendicular to the conductor, there is a force $\boldsymbol{F}$ given by*

$$|\boldsymbol{F}| = |\boldsymbol{B}| \cdot |i| l, \tag{3.42}$$

*where $l$ is the length of the conductor, and the direction of $\boldsymbol{F}$ is perpendicular to the plane of the current and magnetic induction $\boldsymbol{B}$, and for $i$ and $\boldsymbol{B}$ as in Figure 46 the sense of $\boldsymbol{F}$ is toward the reader.*

There is a vector form

$$\boldsymbol{F} = l\boldsymbol{i} \times \boldsymbol{B}$$

of (3.42) for a current vector $\boldsymbol{i}$.

If $\boldsymbol{B}$ is not perpendicular to the conductor, we must use the component of $\boldsymbol{B}$ perpendicular to the conductor in place of $\boldsymbol{B}$ in (3.42). It is to be understood that $i$, as well as $\boldsymbol{B}$, does not change as we go along the conductor.

We suppose that a charge $\Delta q$ passes a section of the conductor in time $\Delta t$ traveling with an average speed $\Delta s/\Delta t$. We picture an element of length $\Delta s$ on the conductor containing the charge $\Delta q$ as shown in Figure 46. The force on the conductor is due only to the moving charges if the conductor is nonmagnetic. If the conductor is magnetic, we restrict ourselves to the force due to the movement of the electric charges. We suppose that $\Delta q$, $\Delta s$, and $\Delta t$ are all positive. By Theorem 3.4 the magnitude of the force acting on $\Delta q$ is $|\Delta \boldsymbol{F}|$, where

$$|\Delta \boldsymbol{F}| = |\boldsymbol{B}|\Delta q \frac{\Delta s}{\Delta t}. \tag{3.43}$$

Dividing by $\Delta s$ and taking the limits as $\Delta t$ goes to zero, assuming that the limits exist, we obtain

$$\left|\frac{d\boldsymbol{F}}{ds}\right| = |\boldsymbol{B}|i. \tag{3.44}$$

Integration yields (3.42). The direction of $\boldsymbol{F}$ follows from that given in Theorem 3.4. The force of Theorem 3.5 is used in electric motors to turn the rotor.

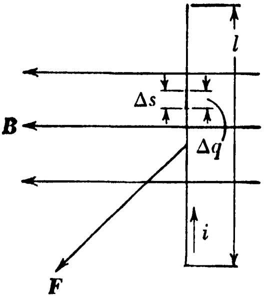

**Fig. 46. Force $F$ on a conductor.**

If $\boldsymbol{B}$ varies in magnitude but not sense from point to point on the conductor, we have

$$|\boldsymbol{F}| = i\int|\boldsymbol{B}|ds \tag{3.45}$$

integrated over the conductor. If at a point on the conductor the sense of $\boldsymbol{B}$ is reversed, the factor $|\boldsymbol{B}|$ in (3.45) must be replaced by $-|\boldsymbol{B}|$. In this case the right member of (3.45) might yield $-|\boldsymbol{F}|$ instead of $|\boldsymbol{F}|$.

If $\boldsymbol{B}$ is in lines per square centimeter, $l$ is in centimeters, and $i$ is in amperes divided by 10, the force $\boldsymbol{F}$ is in dynes.

### 3.8. Faraday's law of induction.

In electric generators used for power the voltage is *induced* in a conductor by moving the conductor in a magnetic field. A law pertaining to this action follows.

***Theorem* 3.6. Faraday's law.** *If a conductor moves in a magnetic field nonvarying with respect to time, there is an electromotive force e generated in the conductor where e is minus the rate at which flux is cut by the conductor with respect to time, that is,*

$$e = -\frac{d\psi}{dt}. \tag{3.46}$$

We consider an element of length $\Delta l$ of a conductor moving with velocity $\boldsymbol{v}$ in a magnetic field with magnetic induction $\boldsymbol{B}$, as shown in Figure 47. We shall suppose that at each point of the element of area $\Delta l \Delta s$, swept over by the conductor element in a time $\Delta t$, the vector $\boldsymbol{B}$ is perpendicular to the surface of the area element in the sense shown. The flux across the area element is $|\boldsymbol{B}|\Delta l \Delta s$ where $|\boldsymbol{B}|$ is evaluated at some point on the area element, on the assumption that $\boldsymbol{B}$ varies continuously over the region of space treated. This flux is the increment $\Delta \delta\psi$ of the flux $\delta\psi$ swept over, that is cut by the conductor element of Figure 47 in moving from some given position, for which we may take $s = 0$, to a value $s$ for the position of the element in the figure. The flux $\delta\psi$ is given by

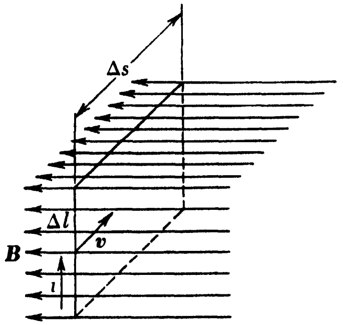

**Fig. 47. Conductor moving in a magnetic field.**

$$\delta\psi = \Delta l \int_0^s |\boldsymbol{B}|ds,$$

it being assumed that the flux always has the same direction across the surface generated by moving the conductor element from $s = 0$ to the position of Figure 47. It follows that

$$\frac{d\delta\psi}{ds} = |\boldsymbol{B}|\Delta l, \tag{3.47}$$

whence

$$\frac{d\delta\psi}{dt} = |\boldsymbol{B}|\Delta l \frac{ds}{dt}. \tag{3.48}$$

Measuring $s$ so that $ds/dt$ is positive, we can write (3.48) as

$$\frac{d\delta\psi}{dt} = |\boldsymbol{B}| \cdot |\boldsymbol{v}|\Delta l \tag{3.49}$$

for the speed $|\boldsymbol{v}|$ of the element. Approximating the conductor by a broken line composed of straight-line elements, and taking the limit as the maximum length of these elements goes to zero, we obtain

$$\frac{d\psi}{dt} = \int_0^l |\boldsymbol{B}| \cdot |\boldsymbol{v}|dl, \tag{3.50}$$

where $\psi$ is the limit of the sum of flux terms of the type $\delta\psi$, the integration being over the conductor.

It is assumed that, as we move a point $P$ along the conductor, the velocity $\boldsymbol{v}$ at $P$, the vector $\boldsymbol{B}$ perpendicular to $\boldsymbol{v}$, and the line $T$ tangent to the conductor at $P$ bear the same sense to each other.

By Theorem 3.4 the force on a unit charge, and hence the electric field intensity $\boldsymbol{E}$, at a point on the conductor element is such that

$$|\boldsymbol{E}| = |\boldsymbol{B}| \cdot |\boldsymbol{v}|. \tag{3.51}$$

The electromotive force $e$ induced in the conductor is the integral

$$e = -\int_0^l |\boldsymbol{E}|dl, \tag{3.52}$$

the direction of $\boldsymbol{E}$ along the conductor being always the same. We can write (3.52) as

$$e = -\int_0^l \boldsymbol{E} \cdot d\boldsymbol{l}.$$

It follows from (3.50)–(3.52) that

$$e = -\frac{d\psi}{dt} \tag{3.53}$$

as was to be proved. The paragraph of (3.54) explains the sign.

If the sense of $\boldsymbol{B}$ is reversed from that shown in Figure 47 we have $-|\boldsymbol{B}|$ in place of $|\boldsymbol{B}|$ in (3.47). We still obtain (3.53). If $\boldsymbol{B}$ is not perpendicular to the surface element of Figure 47, we must replace $\boldsymbol{B}$ in (3.47) by its component perpendicular to this element. If the velocity $\boldsymbol{v}$ of a point on the conductor is reversed relative to $\boldsymbol{B}$, the result is the same as if the sense of $\boldsymbol{B}$ is reversed.

The electromotive force $e$ of one volt is induced in the conductor if $d\psi/dt$ is $10^8$ lines per second, that is, $10^8$ *maxwells* per second, or one *weber* per second.

In the proof of Faraday's law we assumed that the conductor was one-dimensional. This is good enough for practical purposes.

By Theorem 3.4 we know that $\boldsymbol{E}$ is in the direction shown for the induced current $i$ in Figure 47. A statement to this effect follows. See Figure 48.

*A current induced in a conductor moving in a magnetic field is in such a direction that the reaction between the current and the field is such as to produce a force on the conductor tending to oppose the motion of the conductor.*

From Figures 46 and 47 it follows that the current generates a force $\boldsymbol{F}$ opposing the motion of the conductor.

*Fleming's right-hand rule* applies to the current, field, and motion, namely, if the thumb and two adjoining fingers of the right hand are held mutually perpendicular, with the thumb in the direction of the motion, the forefinger is in the direction of the field, and the second in the direction of the curent.

If a current $i$ is maintained in a conductor placed in a magnetic field as in Figure 46, the conductor will move under the influence of the force $F$, but in so doing a *counter electromotive force* will be induced in the conductor so as to oppose the applied voltage. If the resistance of the conductor is $R$, the applied electromotive force is $e$, and the negative of the induced electromotive force is $e_c$, we have

$$i = \frac{e - e_c}{R}. \tag{3.54}$$

Suppose that $e$ and $e_c$ are positive. It follows from (3.54) that for a given $e$, as $e_c$ decreases, the current $i$ increases, whence the corresponding force on the conductor increases in magnitude, and vice versa. In engineering practice we often refer to $e_c$ as the induced electromotive force.

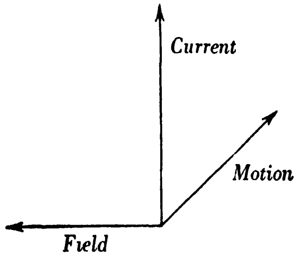

**Fig. 48. Direction of induced current.**

In a later section Faraday's law will be extended to the cases where the field or conductor shape may vary with time.

**3.9. Dipoles.** In treating magnets and magnetic fields it is often convenient to introduce the concept of an *elementary magnet* or dipole. The theory has an exact analogue for electric fields although the concept of the electric dipole is not as useful for our purposes because plus or minus charges can exist alone, whereas from experience we know that the following holds.

*Law* **3.3.** *For a magnetic pole of given intensity $m$ there is always a corresponding pole of opposite intensity $-m$.*

We consider $N$ and $S$ poles a distance $\Delta s$ apart, the $N$ pole being of strength $m$ and the $S$ pole of strength $-m$ as in Figure 49. We let $Q$ denote the midpoint of the line segment joining the poles. We let $P$ be an arbitrary point in space, $\theta$ the angle between the line joining $m$ and $-m$ and the line $QP$ as shown in the figure, while $r_+$ is the distance from $m$ to $P$, and $r_-$ is the distance from $-m$ to $P$, and finally $r$ is the distance from $Q$

to the point $P$. For simplicity we take the permeability of the medium equal to 1. The potential at $P$ due to $m$ is $Km/r_+$ and due to $-m$ the potential at $P$ is $-Km/r_-$ so that the total potential at $P$ due to the two poles is $V$, where

$$V = Km\left[\frac{1}{r_+} - \frac{1}{r_-}\right]. \tag{3.55}$$

Since

$$r_-^2 - r_+^2 = 2r \cos \theta \Delta s \tag{3.56}$$

for the distance $\Delta s$ between the given poles, relation (3.55) is

$$V = \frac{2KMr \cos \theta}{r_- r_+ (r_- + r_+)}, \tag{3.57}$$

where we have written $M$ for the *magnetic moment* $m\Delta s$ of the magnet. We take the limit of $V$ in (3.57) as $\Delta s$ goes to zero while $m$ increases so that $M$ remains constant, and the poles approach $Q$. The angle $\theta$ is kept constant. The potential $V$ of (3.57) becomes

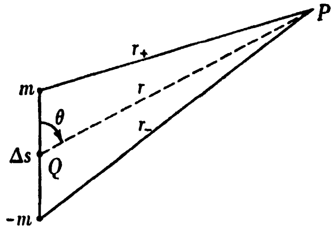

Fig. 49. Elementary magnet.

$$V = \frac{KM \cos \theta}{r^2}. \tag{3.58}$$

By definition $V$ is the potential of a *dipole* at $Q$ with magnetic moment $M$. The south-north direction of the dipole is the same as the direction from $-m$ to $m$ in the original magnet. A magnetic dipole is thus the limit of an elementary magnet as the poles of the magnet approach each other along a straight line while $M$ remains constant.

A magnetic dipole can be conveniently be thought of as a *point magnet*. In many applications, such as a compass needle, the magnet may be thought of as a dipole, the moment $M$ and direction of the dipole being obtained in practice by experiment.

It is sometimes convenient though not necessary to think of some magnetic fields as produced by *magnetic shells*. A magnetic shell is a surface $\Sigma$ for which a magnetic moment density $M$ is associated with each point $Q$ on the surface and a south-north direction along the normal to the surface at $Q$. We consider a region of area $\Delta\Sigma$ about $Q$ as in Figure 50.

We replace this region by a dipole at $Q$ with intensity $M\Delta\Sigma$ and aligned along the normal $N$ at $Q$. We can think of the surface $\Sigma$ as cut up into such elements each of which is approximated by a dipole as described for the element pictured in Figure 50. There is a magnetic field $F_p$ due to the dipoles. We let the areas of the elements go to zero simultaneously by letting the maximum chord joining two points in an element go to zero.

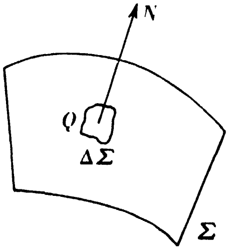

Fig. 50. Magnetic shell $\Sigma$.

If this process yields a unique limiting magnetic field $F$, we say that $F$ is due to a magnetic shell.

Consider a closed circuit in which an electric current $i$ is flowing as in Figure 51. It is supposed that the current has the same numerical value at each point on the conductor and is always in the same direction (clockwise or counterclockwise). It can be shown that at each point not on the shell the magnetic field due to the current is the same as that of a magnetic shell bounded by the conductor. It is understood that the cross section of the conductor is negligible, so that the conductor can be thought of as a curve in space. From the equivalence of the shell to the current, and mathematical theory of integrals, we can show that the field due to the current is conservative in any region not containing the conductor. This can be shown, however, without reference to equivalent shells, the equivalent shell method being at best a device to emphasize the relation between electricity and magnetism.

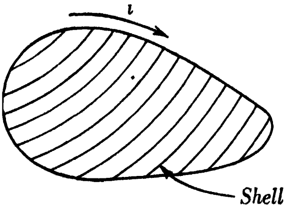

Fig. 51. Circuit with a current $i$.

For the purpose of obtaining a fundamental picture of magnetism it is convenient to treat magnets as bodies made up of dipoles composed of oriented atoms, molecules, and larger subbodies of these magnets. This is not ordinarily necessary for the purpose of setting up engineering problems, involving magnets, in mathematical form.

### 3.10. Magnetic field produced by a single charge and a current in a circuit.

We shall show that the field produced by a single moving charge is *not conservative*.

We shall consider the field with magnetic intensity $\boldsymbol{H}$ at the point $P(x, y, z)$ due to a single positive charge $q$ at the origin $O$ traveling in the positive direction on the $x$-axis with velocity $\boldsymbol{v}$ as shown in Figure 52. Since $\boldsymbol{H}$ is perpendicular to the plane of the $x$-axis and the ray $OP$, we can draw a ray $N$ issuing from $O$ parallel to $\boldsymbol{H}$ and lying in the $yz$-plane. Letting $H_x$, $H_y$, and $H_z$ represent the components of the vector $\boldsymbol{H}$ in the $x$-, $y$-, and $z$-directions respectively, and $\gamma$ the direction angle of $N$ with the $z$-axis, we have

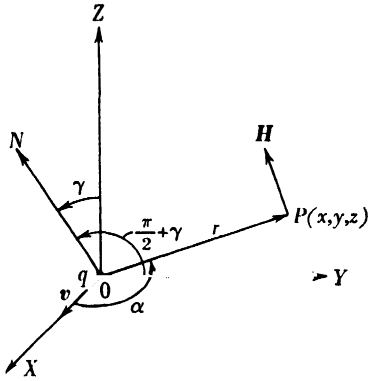

Fig. 52. Magnetic intensity $\boldsymbol{H}$ due to a moving positive charge $q$.

$$H_x = 0, \quad H_y = |\,\boldsymbol{H}\,| \cos\left(\frac{\pi}{2} + \gamma\right) = -\,|\,\boldsymbol{H}\,| \sin \gamma. \tag{3.59}$$

Let $\alpha$ be the angle between the velocity $\boldsymbol{v}$, and the ray $OP$ as in the figure. Since

$$|\,\boldsymbol{H}\,| = \frac{q|\,\boldsymbol{v}\,| \sin \alpha}{r^2}, \quad \tan \gamma = \frac{z}{y}, \quad \sin \alpha = \frac{\sqrt{y^2 + z^2}}{r}, \tag{3.60}$$

we have

$$H_y = \frac{-q|\,\boldsymbol{v}\,|z}{r^3}. \tag{3.61}$$

If we assume that there is a potential function $V$ for the field, the vanishing of $H_x$ implies that $\partial V/\partial x = 0$ and $V$ is a function of $y$ and $z$ only. However, $H_y$ cannot be $\partial V/\partial y$ since $H_y$ is also a function of $x$.

The same type of argument holds for a negative charge at $O$. It follows that *there is not a potential function $V$ for the magnetic field generated by a single moving charge, and hence the field is not conservative in any region of space.*

Although the magnetic field of a single moving charge is not conservative, as remarked in Section 3.9 the field due to a closed circuit (see Figure 51) carrying a current $i$ is conservative in any region of space not containing the conductor. Thus the superposition of the nonconservative fields due to the charges in a closed conductor yields a conservative field for regions not containing the conductor. In contrast, the electric field due to a continuous distribution of charge is a conservative field obtainable by a limit process from the superposition of the conservative electric fields of individual charges. The same remark applies to magnetic poles, and continuous magnetic charge distributions. The proof of the conservativeness of the field due to the current of Figure 51 for regions not containing the conductor is at best complicated, and will be omitted from this section. Once the conservativeness of the magnetic field due to a current carrying conductor is established it is easy to prove that the work done in carrying a charge around a conductor is proportional to the current (see Section 5.1).

We note that the work necessary to move a unit pole at $P$ of Figure 52 around a circle with center on the $x$-axis and in a plane perpendicular to the $x$-axis is the magnitude $|\boldsymbol{H}|$ of $\boldsymbol{H}$ times the circumference of the path. The work done on the pole is thus not zero.

### 3.11. Work done on a conductor moved in a magnetic field.

We shall consider the conductor $C$ of Figure 53 placed in a magnetic field where $\boldsymbol{B}$ designates the component of the magnetic induction (flux density) perpendicular to the surface formed by moving $C$ a distance $\Delta s$ in one direction as in the figure. The density $\boldsymbol{B}$ may vary from point to point on the conductor. We take the current $i$ to be positive. By Theorem 3.5 there is a force $\Delta F$ acting on the element of length $\Delta l$ of the conductor $C$ pictured in Figure 53, where the magnitude of $\Delta \boldsymbol{F}$ is $|\boldsymbol{B}|i\Delta l$, and $\boldsymbol{B}$ is evaluated at a properly chosen point on the element. In moving the element through the distance $\Delta s$ the work done against the force of the field opposing the motion is $|\boldsymbol{B}|i\Delta l\Delta s$ for $\boldsymbol{B}$ evaluated at a properly chosen point on the element of surface generated by moving the element of the conductor. We are assuming here that $\boldsymbol{B}$ varies continuously along the surface, so that on the element of surface the magnitude $|\boldsymbol{B}|$ takes on each value between its maximum and minimum for this element. It follows from limit theory that if the conductor is moved parallel to itself to sweep out a surface $\Sigma$, the work $W$ is given by

$$W = i\iint_{\Sigma} |\boldsymbol{B}|\,dl\,ds = i\iint_{\Sigma} \boldsymbol{B}\cdot d\boldsymbol{\Sigma} \qquad (3.62)$$

integrated over the surface $\Sigma$. By the definition of flux the integral in (3.62) is the total flux $\psi$ across the surface $\Sigma$. Relation (3.62) is valid for any surface generated by moving the conductor and any sense of $\boldsymbol{B}$ with the proper sign in front of $|\boldsymbol{B}|$. We have proved the following, where $\psi$ is said to be the *flux cut by the conductor.*

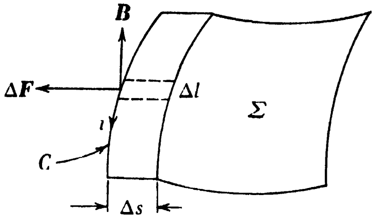

**Fig. 53. Conductor moved in a magnetic field.**

***Theorem* 3.7.** *The work done in moving a conductor, carrying a current i, through a magnetic field is the product of i by the flux $\psi$ cut by the conductor.*

**3.12. Ampere turns.** We shall show why magnetomotive force is often measured in ampere turns.

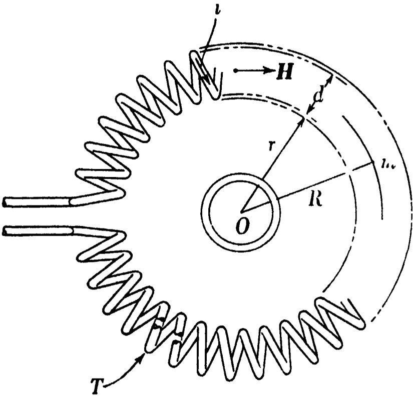

**Fig. 54. Solenoid with $N$ turns.**

As normally used the term *solenoid* refers to a closely wound helical coil of wire. We shall extend the term to cover the coil of Figure 54 wound symmetrically about a circular ring of radius $r$ and cross-sectional diameter $d$. The wire need not be closely wound. The leads in Figure 54, indicating a source of voltage to maintain the current $i$ in the conductor, are neglected in the following argument. The cross section of the wire is assumed to be so small that the wire may be treated as a curve.

We let $T$ designate a single turn of the wire, so that the surface $\Sigma$ of the ring can be generated by moving $T$ about the axis of the ring through the center $O$. We suppose that there are $N$ turns $\{T\}$ composing the coil. We place a magnetic pole with strength $m$ at a point $P$ inside the ring. By Gauss's theorem of Section 3.4, which applies to the surface of the ring, the total flux $\psi$ across the surface $\Sigma$ is $4\pi m$.

We move the pole $m$ about a circle $C$ with center at $O$ perpendicular to the axis of the ring. We consider the work $w$ necessary to bring the pole $m$ once about the circle, and due to the magnetic field of a single turn $T$ only, as in Figure 55. This is the same as the work necessary to move the turn $T$ about the ring keeping the pole $m$ fixed. By Theorem 3.7 this work $w$ is $4\pi mi$. The work of moving the pole $m$ around the ring is thus $4\pi mNi$ for the $N$ turns of wire in the solenoid.

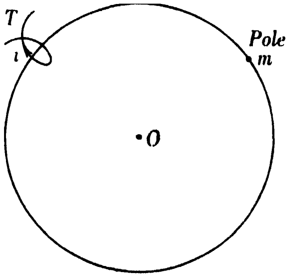

**Fig. 55. Circular path of a pole *m* in a ring.**

Let $C_m$ denote the circle $C$ of radius $r + d/2$ at the middle of the coil. The point $P$ is on a ray perpendicular to $C_m$. Let this ray intersect the wire at a point $Q$. Suppose that the diameter $d$ of the ring is small compared to the radius $r$ of the ring, so that the coil may be treated as a right circular cylinder in a neighborhood of $P$. For each element of current in the coil on one side of $Q$ a distance $f$ along the wire there will be a similar element a distance $f$ from $Q$ on the other side, so that by the Biot-Savart theorem the component of the magnetic field intensity at $P$ perpendicular to $C_m$ vanishes, at least in the limit as $r$ increases beyond all bounds while $d$ and $N/r$ are kept constant. Thus the magnetic intensity $\mathbf{H}$ at $P$ due to the current $i$ may be considered to be tangent to the circle $C$ through $P$. Because of symmetry the magnetic field intensity $\mathbf{H}$ is the same, relative to the coil, at all points of $C$. Let $R$ be the radius of the circle. The work done in moving the pole $m$ about the circle is thus $2\pi R|\mathbf{H}|m$. If the diameter $d$ of the ring section is small compared to the radius $r$ of the ring, the value of $R$ deviates but little from $r$ for the different circular paths in the ring. Since the work $4\pi mNi$ derived above is identical with $2\pi R|\mathbf{H}|m$, it follows that $\mathbf{H}$ is practically the same relative to the solenoid at all points inside of the solenoid (that is, inside the ring). In precise mathematical terms, as $d$ is kept fixed and $r$ is increased beyond all bounds, $N$ being kept proportional to $r$. the limit of $\boldsymbol{H}$ is the same in magnitude at each point in the solenoid.

Any continuous curve $\Gamma$ around the ring within the coil can be approximated by a curve $\Gamma'$ composed of arcs of circles $\{C\}$, and segments of straight lines through the circle $C_m$ perpendicular to $C_m$, and arcs of circles with centers on $C_m$ in planes perpendicular to $C_m$. The integrals of $\boldsymbol{H} \cdot d\boldsymbol{s}$ along the arcs of the circles with centers on $C_m$ will vanish, as will these integrals along the segments of straight lines through $C_m$, because $\boldsymbol{H}$ is perpendicular to these arcs. Thus when $r$ is large compared to $d$ the integral of $\boldsymbol{H} \cdot d\boldsymbol{s}$ around any curve $\Gamma$ will be $2\pi r|\boldsymbol{H}|m$ for practical purposes, and thus the work done in bringing a pole $m$ once around the ring is the same for all paths.

We write $l$ for the length of a circular path around the solenoid, so that

$$|\boldsymbol{H}| = \frac{4\pi Ni}{l}. \tag{3.63}$$

The *magnetomotive force* $\mathfrak{F}$ is the work $l|\boldsymbol{H}|$ per unit pole necessary to bring a magnetic pole around the interior of the solenoid. The magnetomotive force $\mathfrak{F}$ due to $i$ amperes and $N$ turns is said to be $Ni$ *ampere turns*. The *average value* of the field intensity $\boldsymbol{H}$, termed the magnetizing force, is then measured as to magnitude in *ampere turns per inch*.

Due to symmetry the magnetomotive force $\mathfrak{F}$ for a partial path in the solenoid is the ampere turns per inch times the number of inches along the solenoid taken up by the path. For a solenoid with $d$ small compared to $r$, the part of the ring between two cross sections made by planes through the axis of the ring, which axis passes through $O$, can be considered to be a straight coil wound on a right circular cylinder, provided the sections are not too far apart. Further, the field at one section of the ring depends primarily on neighboring turns, the distant ones having little influence. *It is therefore reasonable in practical problems involving solenoids to suppose that these solenoids are parts of solenoidal rings of the type we have been treating.* We may then consider that the field intensity is constant in the interior of the solenoid, and the magnetomotive force may be taken in ampere turns. Results accurate to one or two per cent may be expected in many engineering examples.

The solenoid of Figure 54 was taken with circular cross section. The above development holds regarding $\mathfrak{F}$, however, for a ring with constant cross section bounded by *any* simply closed *curve*. In practical engineering examples considerable deviation from constant cross section may be made without appreciable error. Unless there are definite reasons for not doing so, the reader should assume that the above treatment applies to his case.

As an illustration of the practical use of ampere turns we consider an ordinary toroidal ring of armco iron with a 0.01-inch air gap. We let the length of an average path around the ring be 10 inches, and the cross-sectional area of the core be 2 square inches. Suppose that a core flux of 129,000 lines per square inch is desired. Since the core area is 2 square inches the flux density is 64,500 lines per square inch. A typical magnetization curve for armco iron shows that a field intensity of 5 ampere turns per inch is required, so that for a 10-inch path a magnetomotive force of 50 ampere turns is needed.

For *air* the field intensity $H$ in ampere turns per inch is approximately $\frac{4}{13}$ times the flux density in lines per square inch. For 64,500 lines per square inch we obtain 20,000 ampere turns per inch. The magnetomotive force required for the air gap is thus 200 ampere turns.

The total magnetomotive force required for the ring is 250 ampere turns.

In practice the cross-sectional core area at the gap is increased to take care of fringing and leakage. This increase is normally less than 20 per cent.

### 3.13. Capacitors.

Two surfaces of conductors separated by a dielectric (or by a vacuum) form a *capacitor.* A capacitor was formerly called a *condenser.* Suppose that all points on one such surface $\Sigma$, as in Figure 56, are not at the same potential $V$. We suppose that $V$ and the derivatives of $V$ needed for the following argument vary continuously over $\Sigma$. There is then a point $P_1$ on $\Sigma$ where the directional derivative $dV/ds$ for some direction does not vanish. It follows that there is a nonzero component $E_s$ of the electric field intensity at $P_1$. We can then choose a neighborhood $n_1$ of the point $P_1$ on $\Sigma$ so that the component $E_s$ has everywhere a magnitude greater than some positive number $a$. There is thus a net force on the charge in $n_1$ which will cause this charge to move so that there is a current on the surface. In practical examples we may suppose that there is always charge in $n_1$ on which the force can act. Further we can associate a *resistance* $R$ with neighborhoods of each pair of points $P_1$ and $P_2$ on $\Sigma$ in the following manner. We construct a neighborhood $n_2$ about $P_2$. Suppose that each point in $n_1$ is at the same potential, and the same is true of points in $n_2$. By Ohm's law we obtain a resistance $R$ between $n_1$ and $n_2$.

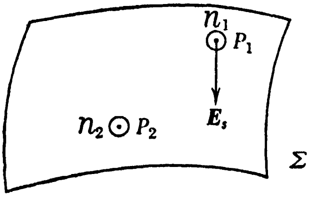

Fig. 56. Points on a charged surface.

In any practical example of a conducting surface used in a capacitor, the resistance between any two points on the surface is relatively small, so that for a small difference of potential between two points $P_1$ and $P_2$ on the surface there is a large current $i$ flowing between these points, or more correctly between the neighborhoods of these points.

We suppose that no potential difference is maintained between points on the conducting surface $\Sigma$ by external means. If potential differences persist between points on $\Sigma$ so that the power $dW/dt$ as given in (3.38) always exceeds a positive number $n_0$, we can draw any amount of energy from the surface over a sufficiently long period of time, and thus have a case of "perpetual motion." This contradicts the assumption, to be treated in the chapter on thermodynamics, to the effect that this is impossible. Thus the surface will come to an equilibrium so that the potential is the same at all points on the surface.

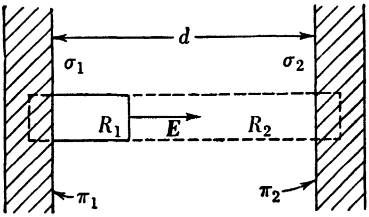

**Fig. 57. Parallel plane conducting plates.**

For capacitors used in practice, the potential becomes uniform over each surface in much less than a microsecond (one millionth of a second), so that this process may be assumed to take place instantly.

If charges of like sign are placed in a conductor in the case where potential differences are not externally maintained, repulsion between these charges will force them to the surface of the conductor.

We consider two large plane conducting plates parallel to each other as in Figure 57. By a *plane plate* we mean a conductor bounded by two parallel planes relatively close together. *Close together* signifies that the thickness of a plate is small compared to all other linear dimensions of the plate.

We let $\pi_1$ be the surface of the first plate facing the second plate, and $\pi_2$ the surface of the second plate facing the first plate. We suppose that each of these surfaces is equipotential. The electric field intensity is then normal (outwardly) to the surface, and the same is true of the flux lines. One would expect that, for practical purposes, there is then a uniform charge density $\sigma_1$ on $\pi_1$, and a similar charge density $\sigma_2$ for $\pi_2$. This can be proved from the theory of partial differential equations.

If the surfaces $\pi_1$ and $\pi_2$ are close together we may treat them for the moment as if they were infinite in extent, that is, planes. Due to sym-

*metry the flux lines run from $\pi_1$ to $\pi_2$ along* straight lines perpendicular to these surfaces. At each point between these surfaces the electric field intensity $E$ is perpendicular to these surfaces. We construct a cylindrical surface with elements perpendicular to the given planes $\pi_1$ and $\pi_2$, and cut off a region $R_1$ bounded by this surface and two planes parallel to $\pi_1$, one inside the first plate and the other between these plates. Let $A$ be the cross-sectional area of the cylinder. By Gauss's theorem (Theorem 3.3) the flux across the surface $\Sigma$ bounding $R_1$ is $4\pi\sigma_1 A$. There is no flux across the part of $\Sigma$ inside the plate because the field intensity vanishes at each point within the plate. There is also no flux across the given cylindrical surface whose sides are lines of flux. It follows that the total flux is across the face of $R_1$ nearest the plane $\pi_2$. Using this same face for a region $R_2$ formed with respect to $\pi_1$ and $\pi_2$ just as $R_1$ was constructed with respect to $\pi_2$ and $\pi_1$, we find that the flux across this face is $4\pi\sigma_2 A$. Since the outward normal to $R_1$ on this face has the opposite sense of the outward normal to $R_2$ on this same face it follows that

$$\sigma_2 = -\sigma_1. \tag{3.64}$$

Let us suppose that $\sigma_1$ is positive. The magnitude of the electric induction $D$ at each point between the planes $\pi_1$ and $\pi_2$ is given by

$$| D | = 4\pi\sigma_1. \tag{3.65}$$

By (3.26) and (3.65) the magnitude of the electric field intensity $E$ is given by

$$| E | = \frac{4\pi\sigma_1}{k} \tag{3.66}$$

at each point between the given planes, $k$ being the dielectric constant of the medium between the plates.

The above argument involving $R_1$ and limit methods may be used to show that in general

$$\text{Lim } | E | = \frac{4\pi| \sigma |}{k} \tag{3.67}$$

for the limit of the electric field intensity magnitude $| E |$ along a flux line to an equipotential surface $\Sigma$ as a point $P$ on $\Sigma$ is approached, $\sigma$ being the charge density at $P$, and $k$ the limit of the dielectric constant of the medium as $P$ is approached along the flux line. In usual terminology the vector $E$, whose magnitude is given by (3.67), is the *electric field intensity just off the surface* $\Sigma$.

It can be shown by elementary limit methods that the quantity $E$ at each point on the surface $\Sigma$ is one-half that given in (3.67). Thus the electric *field intensity* is quite *discontinuous* at the surface of a conductor in static equilibrium, being zero inside the conductor, having magnitude $2\pi| \sigma |/k$ on the surface, and $4\pi| \sigma |/k$ just off the surface.

The above discussion of the surface $\Sigma$ of a conductor is based on certain logical and simple assumptions about mathematical limits, and continuity of the charge density $\sigma$ on $\Sigma$.

The potential difference $e$ between the plates of Figure 57 is the work done in bringing a unit positive charge from one inner surface to the other. Letting $d$ be the distance between the plates we have

$$e = \frac{4\pi\sigma_1 d}{k}. \tag{3.68}$$

The field intensity $E$ is away from the plane $\pi_1$ since $\sigma_1$ is assumed to be positive. Following convention we shall say that $e$ is the potential at $\pi_1$ minus that at $\pi_2$. In fact, if we imagine the plates connected by a conductor, at the instant of connection the positive charges on $\pi_1$ will tend to travel to $\pi_2$.

For plane capacitor plates used in practice the cross section is small enough compared to the other dimensions so that these plates may be considered to be parts of large plane plates, in which case the potential difference between them is given by (3.68). If $S$ is the area of the inside surface of such a plate the total charge $q$ on the plate is given by

$$q = \sigma_1 S. \tag{3.69}$$

It follows from (3.68) and (3.69) that

$$q = Ce \tag{3.70}$$

for a constant $C$. The coefficient $C$ of proportionality in (3.70) is called the *capacitance* of the capacitor.

In the usual engineering units $e$ is in volts and $q$ in coulombs, whence $C$ is in *farads*. The farad is too large a unit to use in practice, and for this reason the *microfarad* is employed, this being one millionth of a farad and abbreviated $\mu f$. A millionth of a microfarad is termed a micromicrofarad, written $\mu\mu f$.

Capacitors made of metal plates separated by air usually cover a range from 3 to 1,000 micromicrofarads.

If greater capacitances are desired the dielectric can be changed. Capacitors of mica are often used for capacitances from 50 micromicrofarads to 0.1 microfarad. Impregnated paper capacitors normally cover a range from 0.0001 to one microfarad. Electrolytic capacitors are often used for the range from one to 1,000 microfarads. It is to be recalled that an *electrolyte* is a conducting solution.

For normal ranges from zero to a few hundred volts the above capacitors can be obtained in small sizes occupying no more than a few cubic inches and weighing no more than a few ounces. See Figure 58.

Dielectrics used in capacitors can conduct a little current, so that some current leaks through a capacitor if there is a potential difference between the leads. When the voltage is sufficiently high the dielectric breaks down and becomes a conductor of comparatively low resistance. This voltage varies with the capacitor. Thus for many radio capacitors this voltage ranges from 100 to 1,000 volts. When an arc forms between two electrodes in air the resistance between these electrodes through the air becomes very small, so that only a small voltage is necessary to maintain a large current.

The capacitance of a paper capacitor will vary from time to time because of changes in the dielectric constant k, but will ordinarily be constant to 1 per cent The capacitances of mica capacitors often are constant to 0.025 per cent, whereas those of electrolytics will often vary by 100 per cent.

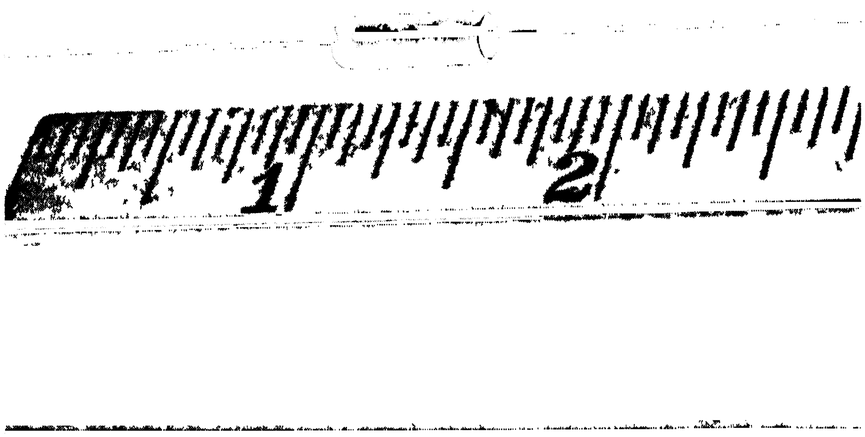

**Fig. 58. A midget industrial condensor. Courtesy of the Cornell-Dubilier Electric Corporation.**

The capacitance of a capacitor will change with temperature, frequency of applied alternating voltages, and other factors.

The conductor surfaces in a capacitor need not be a plane, in fact, will often not be so. The above treatment for parallel plane plates when applied to other surfaces not far apart, such as right circular cylinders, shows that (3.70) applies.

As will be seen later, several capacitors may be coupled to give a resultant capacitor with much larger capacitance. This is actually done in commercial products, such as variable air capacitors with several plates.

**3.14. An induction law.** We shall extend the treatment of Section 3.8 to the general case of varying flux.

We consider the conductor element of length $\Delta l$ of Figure 47. We may picture the ends of this element as moving on rails as shown in Figure 59. The element need not be moved parallel to itself. In Figure 59 the magnetic flux density normal to a surface $\Sigma$ through the circuit of this figure is denoted by $\boldsymbol{B}$. We let $\psi$ designate the total flux across this surface so that

$$\psi = \iint_{\Sigma} |\boldsymbol{B}| d\Sigma \tag{3.71}$$

integrated over $\Sigma$, it being supposed for the moment that the vector $\boldsymbol{B}$ is in the direction of the normal to $\Sigma$. We can write (3.71) in vector form as

$$\psi = \iint_{\Sigma} \boldsymbol{B} \cdot d\boldsymbol{\Sigma}.$$

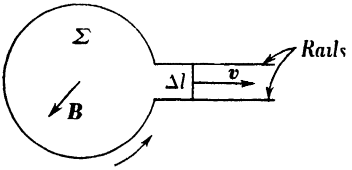

**Fig. 59. Conductor element of length $\Delta l$ moving in a magnetic field.**

The electromotive force $e$ generated in the moving element is then given by (3.46). Since $\psi$ in (3.71) is the flux across a surface bounded by the conductor, the following theorem is suggested.

***Theorem* 3.8.** *Let $\psi$ denote the magnetic flux across a surface bounded by a closed circuit $C$ with a continuously turning tangent, and suppose that the magnetic field intensity varies continuously (as a function of position) over the region of space considered. For the rate of change $d\psi/dt$ of the flux with respect to time $t$ there is an electromotive force $e$ for $C$ where*

$$e = -\frac{d\psi}{dt}. \tag{3.46}$$

The flux $\psi$ can change in two ways. One of these is by deformation of the conductor from one shape $C_1$ to another shape $C_2$ as in Figure 60 while the magnetic field remains constant. The other is where the conductor remains constant in shape while the magnetic field changes. Of course, both phenomena may occur at the same time. Because of the principle of superposition of forces each effect can be considered separately. It is to be remembered that $e$ is the integral of the tangential component of electric field intensity around the conductor, and that this intensity can be thought of as the sum of two intensities, one due to the rate of change of the magnetic field, and the other the rate of change of conductor shape.

Consider first a constant magnetic field but a changing conductor shape as in Figure 60. In going from the shape $C_1$ to the shape $C_2$ in a time $\Delta t$ we may think of this as the limit of a sequence of steps, in which elements of $C_1$ such as the element of length $\Delta l$ of Figure 60 are separately moved over rails to new positions occupying each at least one point on the curve $C_2$ as shown in Figure 61. In Figure 61 sections of the curve $C_1$ comprising $C_1$ have been moved to new positions. Each element is moved in such a way that the ends of the element describe parallel lines, and one of these lines is normal to the curve $C_1$. After this operation the side $AB'$ of a tooth is pivoted about the point $A$ on $C_1$ to the position $AB''$ on the line $AB$ forming a side of the next tooth as shown in Figure 62. This is done for each tooth so that from the curve $C_1$ there is obtained a curve $C_1^*$ of which a section $CB'B''ABD'D''ED$ is shown in Figure 63.

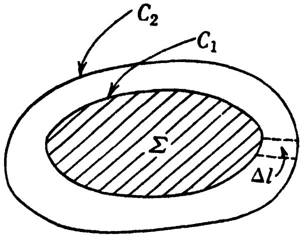

**Fig. 60. Conductor $C_2$ obtained by the deformation of the conductor $C_1$.**

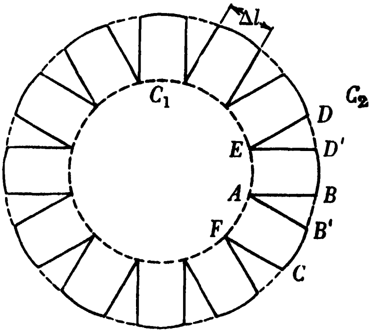

**Fig. 61. Elements of the conductor $C_1$ moved to $C_2$.**

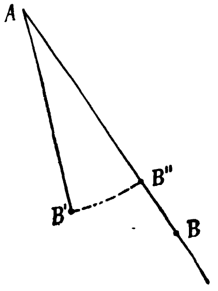

**Fig. 62. Closing of the triangular sections of Fig. 61.**

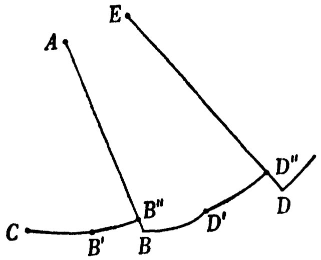

**Fig. 63. Section of a curve $C_1{}^*$ whose emf approximates that of $C_2$.**

In Figure 63 the arc $CB'$ is the new position of an element $FA$ on $C_1$ in Figure 61. The arc $B'B''$ is obtained by the pivoting operation pictured in Figure 62. The line segment $B''A$ lies on the line segment $AB$, or $AB$ extended. The arc $BD'$ is the new position of the next element $AE$ on $C_1$, etc. The electromotive force from $B''$ to $A$ is the negative of that from $A$ to $B''$, so that these cancel in obtaining the electromotive force about the curve $C_1^*$. Thus the electromotive force about $C_1^*$ is the work done in bringing a unit positive charge around the path $CB'B''BD'D''D$, etc. Let this path be denoted by $C_2^*$. For the problems arising in engineering we may suppose that as the largest of the lengths $\Delta l$ of the elements of arc on $C_1$ go to zero, the curve $C_2^*$ approaches $C_2$ uniformly, and the electromotive force for $C_1^*$ approaches the electromotive force for $C_2$. In going from $C_1$ to $C_2$ we have cut a certain flux $\Delta\psi$. We can think of each of the above operations described for going from $C_1$ to $C_1^*$ as occurring in the time $\Delta t$, so that by Faraday's law there is an *average* electromotive force for $C_1^*$ and the time $\Delta t$, equal to the flux cut in going from $C_1$ to $C_1^*$ divided by $\Delta t$, and in the limit as the $\Delta l$'s go to zero, there is such an average electromotive force $-\Delta\psi/\Delta t$ for $C_2$.

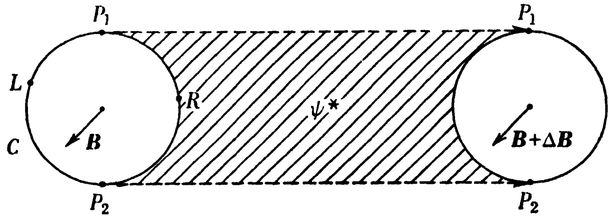

Fig. 64. Plane circuit moved in a plane by translation.

We have to use the word *average* because we have not yet taken the limit as $\Delta t$ goes to zero. Taking the limit as $\Delta t$ goes to zero we get an electromotive force $e$ in $C_1$ given by (3.46).

To complete the proof of Theorem 3.8 we must treat the case of a constant conductor in a varying magnetic field. We consider the special case of a plane closed circuit $C$ as shown in Figure 64. In this figure the magnetic field intensity normal to the plane within $C$ at a given instant $t$ is designated by $\boldsymbol{B}$. We let $(\boldsymbol{B} + \Delta\boldsymbol{B})$ designate the new value of the magnetic field intensity at the same point at the instant $(t + \Delta t)$. Instead of keeping the conductor fixed in the plane and changing the field intensity from $\boldsymbol{B}$ to $(\boldsymbol{B} + \Delta\boldsymbol{B})$, we may think of moving the conductor $C$ to a new position in the plane by translation, where in the new position the normal component of magnetic field intensity is given by $(\boldsymbol{B} + \Delta\boldsymbol{B})$. Of course, we are making the reasonable assumption that the magnetic field intensity varies continuously (as a function of position) within $C$ in each of its initial and final positions. By having the left and right positions of $C$ in Figure 64 separated by intervening space we may suppose that the field traversed by $C$ in going from the left position to the right varies continuously from point to point (varies with position, not time).

In Figure 64 the point $P_1$ is a highest point on the curve $C$, while $P_2$ is a lowest, and $C$ is moved horizontally in going from one position to the other. The flux $\psi^*$ across the shaded region in the figure is cut by both the arc $P_2RP_1$ and $P_2LP_1$. We let $\psi$ be the flux across the region bounded by $C$ in its left position and $(\psi + \Delta\psi)$ the flux for the right position. The flux cut by the arc $P_2RP_1$ in the time $\Delta t$ is $(\psi + \Delta\psi + \psi^*)$, while the flux cut by the arc $P_2LP_1$ in the same time is $(\psi + \psi^*)$. It follows from Faraday's law that there is an *average* electromotive force $-\Delta\psi/\Delta t$ generated in the circuit $C$. We note that to get the average electromotive force around $C$ we subtract this electromotive force for the arc $P_2LP_1$ from the electromotive force for the arc $P_2RP_1$. Taking the limit as $\Delta t$ goes to zero, we have (3.46) for the case of flux varying across the region of the plane bounded by a plane closed circuit. The results hold for rather general plane circuits.

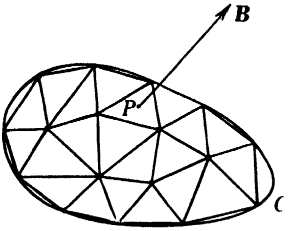

**Fig. 65. Surface bounded by a conductor C, approximated by a surface of triangles.**

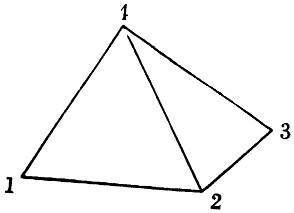

**Fig. 66. Surface formed by two triangles bounded by conductors.**

We consider now any closed curve $C$ in space as in Figure 65. Let $\Sigma$ be a surface bounded by $C$. It is assumed that there is a normal at each point on $\Sigma$ and a normal magnetic field intensity $\boldsymbol{B}$. We replace $\Sigma$ by a surface $\Sigma^*$ made up of triangles as shown in Figure 65, such that each triangle is tangent to the surface $\Sigma$ at one or more points $\{P\}$ and the vertices on the edge of $\Sigma^*$ lie on $C$.

We shall treat the electromotive force around the boundary of a surface formed by two triangles as in Figure 66. We let $e_{ij}$ be the work necessary to bring a unit positive charge from the point $i$ to the point $j$ in this figure. If the flux across the surface is changing there is an electromotive force $e_1$ generated in the circuit 124 equal to the rate of change $-d\psi_1/dt$ of the flux $\psi_1$ across the triangle bounded by this circuit, and similarly an electromotive force $e_2$ generated in the circuit 234 equal to $-d\psi_2/dt$ for the flux $\psi_2$ across the triangle bounded by this circuit. The work done to bring a unit positive charge from the point 4 to the point 2 is the negative of that necessary to bring this charge from the point 2 to the point 4. It follows that the electromotive force generated around the path 12341 is the sum $(e_1 + e_2)$, which is minus the derivative $d(\psi_1 + \psi_2)/dt$ of the flux $\psi_1 + \psi_2$ across the entire surface formed by the two triangles.

The argument for two triangles clearly extends to any number of triangles. We let $C^*$ be the boundary of the surface $\Sigma^*$ of Figure 65. The electromotive force $e^*$ around $C^*$ is $-d\psi^*/dt$ for the flux $\psi^*$ across $\Sigma^*$. We form a sequence of surfaces $\{\Sigma^*\}$ continuously deforming into $\Sigma$, and take the limit $e$ of the corresponding sequence $\{e^*\}$, assuming that it exists, and also the limit $\psi$ of the $\psi^*$'s, the quantity $\psi$ being the flux across $\Sigma$. It follows that (3.46) applies to the general case of flux varying across the surface bounded by the circuit $C$, whence Theorem 3.8 is proved.

It might be thought that the flux $\psi$ of Theorem 3.8 depends on which surface $\Sigma$ is taken to be bounded by the curve $C$. This is not so since the flux across the surface formed by two such surfaces $\Sigma_1$ and $\Sigma_2$, and bounding a simply closed region devoid of charge, is zero by Gauss's theorem. It follows that the flux across $\Sigma_1$ is equal to the flux across $\Sigma_2$ for the proper normal directions to these surfaces.

We have considered *three cases of induced electromotive force.* They are as follows:

***Case* 1.** *The conductor shape and the field may be nonvarying with respect to time, while the conductor moves as a rigid body relative to the field.* By Faraday's law of Section 3.8 we have an induced electromotive force corresponding to the rate $d\psi/dt$ at which flux $\psi$ is cut by the conductor. This flux $\psi$ is not to be confused with the flux across a surface bounded by the conductor as in the treatment of Theorem 3.8. The flux discussed in Section 3.8 is the flux across a surface generated by the conductor in its motion.

***Case* 2.** *The conductor shape may be nonvarying while the field varies with respect to time, there being no motion of the conductor as a rigid body.* This is covered by Theorem 3.8.

***Case* 3.** *The conductor shape may be varying with respect to time, while the field is nonvarying, and there is no motion of the conductor as a rigid body.* In this case we may think of the deformation of the conductor as motion relative to the field so that this case is related to Case 1. Theorem 3.8 applies to this deformation.

When two or all of the processes are occurring simultaneously we add the corresponding electromotive forces.

In Theorems 3.6 and 3.8 it is necessary to adopt the point of view of relativity in that *we must treat our fields and motions with respect to a given coordinate system.* Thus suppose that we have a field unchanging with respect to an $xyz$-coordinate system. Let a conductor be moved relative to this coordinate system, so that an electromotive force is generated in the conductor. *If the conductor is shielded by an iron tube rigidly attached to the conductor, and the conductor undergoes the same motion as before, the same electromotive force is induced in the conductor as for the motion without the shield.* This is true even when the total magnetic intensity is zero at the conductor, and the shielded conductor does not cut lines of flux due to the resultant field. The conductor does not move relative to the shield, and there is therefore no electromotive force induced in the conductor because of the presence of the field set up by the shield. The electromotive force is thus due to the given field only.

If the iron is not saturated, enough of the elementary magnets in the iron line up so that the magnetic field of these magnets neutralizes the given field. If the iron is saturated, and the given field is strong enough this neutralization will not be possible.

We shall express the principal results of this section in vector notation. Consider a given coordinate system $S_1$, and let $\boldsymbol{B}$ denote the magnetic intensity of a field relative to this coordinate system. Suppose that the magnetic field is varying. By Section 5.1 to follow the electric intensity $\boldsymbol{E}$ relative to $S_1$, and associated with the given magnetic field, satisfies Maxwell's equation

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

Let $\boldsymbol{v}$ be the velocity of a charged particle relative to $S_1$, and let $S_2$ be a coordinate system attached to the particle. By Theorem 3.4 the force intensity $\boldsymbol{E}'$ acting on the particle is

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}.$$

Let $C$ designate a conductor (this does not have to be closed). The emf induced in $C$ due to a field of intensity $\boldsymbol{E}'$ is the line integral

$$\int_C (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{s}.$$

Suppose now that $C$ bounds a surface $\Sigma$ to which Stokes's theorem applies. By this theorem

$$\int_C (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{s} = -\iint_\Sigma \left(\frac{\partial \boldsymbol{B}}{\partial t} - \operatorname{curl} \boldsymbol{v} \times \boldsymbol{B}\right) \cdot d\boldsymbol{\Sigma}.$$

With some manipulation the last integral can be seen to reduce to

$$-\frac{d}{dt}\iint_\Sigma \boldsymbol{B} \cdot d\boldsymbol{\Sigma}.$$

**3.15. Self-inductance.** We shall consider a solenoid with $N$ turns carrying a current $i$ as in Figure 67. We shall simplify our discussion by assuming first that all turns occupy the same space. By the Biot-Savart theorem and integration over the coil, assuming for mathematical purposes that the cross section of the wire is zero, we find that the magnetic field intensity at each point on the coil due to a single turn is proportional to the current $i$. Due to $N$ turns this intensity is proportional to $Ni$.

If the current $i$ varies, the magnetic field due to the current $i$ changes. By Theorem 3.8 there is a *counter electromotive force* in each turn due to this effect. We assume that the medium is of constant permeability $\mu$. Since the magnetic induction $\boldsymbol{B}$ is proportional to the magnetic field intensity $\boldsymbol{H}$ at each point, it follows that $\boldsymbol{B}$ is proportional to $Ni$. By Theorem 3.8 the electromotive force in a single turn, due to the magnetic field of the current $i$, is proportional to the derivative $dNi/dt$. Since there are $N$ turns the total electromotive force $e$ is proportional to $N^2di/dt$, that is

$$e = -L\frac{di}{dt} \tag{3.72}$$

for a constant $L$. The quantity $L$ is termed the *inductance* of the solenoid. Since $L$ is due to the solenoid itself it is often called the *self-inductance* of the coil. For the simple coil we have been treating the inductance $L$ is *proportional to the square of the turns.*

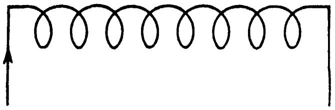

**Fig. 67. Solenoid.**

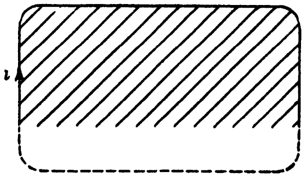

**Fig. 68. Surface obtained by unraveling the coil of Figure 67.**

In a practical case all turns do not occupy the same space. The relation (3.72) still applies since the flux density at each point is proportional to the current $i$. To attain a given value of $L$ it is necessary to use more than the $N$ turns which will be satisfactory for the simple coil.

The coil of Figure 67 is not a closed circuit. However, we can unravel this coil to obtain the wire of Figure 68. The coil is usually so far away from most of the rest of the circuit that the flux density everywhere except in the immediate vicinity of the coil can be neglected. We can form a surface through this unraveled circuit shown in Figure 68. A surface through this circuit can be deformed into a surface through the coil, as the wire of Figure 68 is bent into the shape of the coil. The flux *linking* the coil, and used in applying Theorem 3.8 here, is understood to be the flux across such a surface.

As was the case for the counter electromotive force $e_c$ of a conductor moving in a magnetic field, occurring in (3.54), the same argument shows that the electromotive force $e$ of (3.72) is such as to *oppose the change in i.* Thus if the electromotive force applied to the coil externally to maintain the current $i$ is increased to increase $i$, the counter electromotive force $e$ of (3.72) will oppose the applied electromotive force so that the net electromotive force is the sum of these quantities, $e$ being negative.

For a wire not wound in the form of a coil the inductance $L$ of the wire is normally relatively small compared to its resistance $R$.

If $i$ is in amperes, time $t$ is in seconds, and $e$ is in volts, the inductance $L$ is in *henries*. If the medium is air the coil is said to be *air-cored;* with iron, *iron-cored*. Normal-sized air-cored coils, occupying not more than one or two hundred cubic inches, cover an inductance range from 0.1 *millihenry* (0.0001 henry) to 1 henry. By using iron cores instead of air the flux density $B$ is increased by a certain factor $\mu$, so that coils weighing only three pounds and possessing an inductance of 1,000 henries are available. Unless powdered iron is used for the core the permeability $\mu$ in the core will vary greatly, with wide variations of $L$ resulting. Coils with powdered iron cores weighing only a few ounces, occupying only two or three cubic inches, and having an inductance of 0.1 henry are not uncommon, especially in voice transmission work. See Figure 69 picturing an inductor standard.

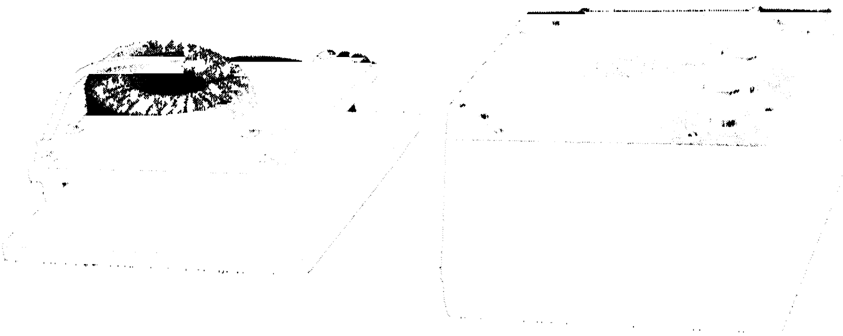

**Fig. 69. An inductor standard of 0.1 millihenries. Courtesy of the General Radio Company.**

The wire in a coil has a resistance $R$ which usually cannot be neglected. Rarely ever does the ratio $L/R$ attain the value 10. In the case of industrial coils, often called *chokes*, the resistance $R$ of the coil is frequently 100 or 1000 times as much as the inductance $L$.

The inductance $L$ will vary with the frequency of the current if alternating current is used, this variation being due to a different distribution of the current in the wire thus producing a different flux distribution. The inductance $L$ will also change with variations in the permeability of the medium, and other factors.

**3.16. Mutual inductance.** If a current $i_1$ flows through a coil $C_1$, and there is a second coil $C_2$ near $C_1$, it may be necessary to consider the effect of the magnetic field, due to $C_1$, on the coil $C_2$. By Theorem 3.8 there is an electromotive force $e_2$ in coil $C_2$ where

$$e_2 = -M \frac{di_1}{dt} \tag{3.73}$$

for a constant $M$, the flux due to $C_1$ and "linking" $C_2$ being proportional to the current $i_1$ in $C_1$. Two such coils are pictured in Figure 70. There is also an electromotive force $e_1$ in $C_1$ given by

$$e_1 = -M^* \frac{di_2}{dt}. \tag{3.74}$$

We shall show that $M$ and $M^*$ are equal.

The flux linking the coil $C_2$ due to the current $i_1$ in the coil $C_1$ is $Mi_1$, and the flux linking the coil $C_1$ due to the current $i_2$ in the coil $C_2$ is $M^*i_2$. There is a net force $F_2$ acting on $C_2$ due to the magnetic field of $C_1$ and a force $F_1$ acting on $C_1$ due to the magnetic field of $C_2$. By Newton's action-reaction law $F_1 = -F_2$. We remove the coil $C_2$ from the influence of the coil $C_1$ so that the flux from $C_1$ linking $C_2$ is zero, and then bring the coil $C_2$ into the original position of $C_2$ relative to $C_1$. If $C_2$ were a single turn we could move it parallel to itself as was done with $C$ in Figure 64. In any case the net flux cut by $C_2$ is $Mi_1$. By Theorem 3.7 the work done in bringing $C_2$ back is $Mi_1i_2$. Instead, we could have thought of this operation of bringing $C_2$ back to $C_1$ as one where $C_1$ is brought back to $C_2$, in which case the work done is $M^*i_2i_1$. It follows that the $M$'s are equal, whence there is a unique coefficient $M$ of *mutual induction* between the two coils.

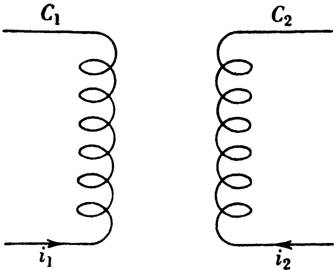

**Fig. 70. Coils $C_1$ and $C_2$ inductively coupled.**

The values of $M$ attainable for reasonably sized coils are the same as those for $L$ discussed in Section 3.15. Mutual inductance is particularly important for *transformers* of all kinds, since conductively distinct circuits may be coupled inductively.

The constant $M$ depends, as does $L$, on the geometry of the coils, and can be varied over wide ranges by changing the relative positions of the coils, as in *variometers*.

**3.17. Networks.** *The standard electric circuit is composed of* **resistors,** capacitors, and inductors as elements. We usually think of a circuit as a simply closed wire. We may, however, have a *network* of such wires as shown in Figure 71 composed of more than one simply closed circuit. The points $P_1, \ldots, P_7$ of Figure 71 are called *branch points,* or *nodes,* and the sections of wire between consecutive branch points are termed *branches.* The network of Figure 71 has 11 branches.

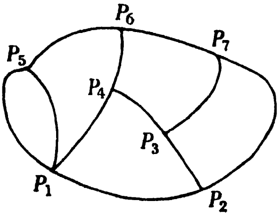

**Fig. 71. Network.**

A branch need not be a continuous wire. It may be a wire joining batteries, capacitors, and other electric elements to be taken up later. Although a capacitor inserted in a branch breaks up the conductor continuity, we can think of a current as traversing the space between conductor plates (see Section 5.1).

The treatment of electric networks is much simplified by the use of notations for electric elements, as in Figure 72. These need not correspond to different physical bodies. Thus a coil is pictured in Figure 73 as a resistance $R$ followed by an inductance $L$, the quantities $R$ and $L$ being the resistance and inductance of the coil, respectively. When one electric element is followed in a branch by another as in Figure 73 we say that these elements are in *series.* That the arrangement of Figure 73 is justified is seen from the fact that there is a potential difference $Ri$ across the coil due to the resistance $R$ and current $i$, and a potential difference $Ldi/dt$ because of the inductance of the coil. These potential differences are clearly additive, so that the total potential drop across the coil is

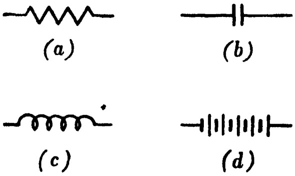

**Fig. 72. Symbols for electric elements.**
**a. Resistance.**
**b. Capacitance.**
**c. Inductance.**
**d. Battery.**

$(Ri + L\,di/dt)$. This is precisely the drop across the two elements in Figure 73.

There is also capacitance between the turns of the coil, two such turns forming an elementary capacitor. If we take into account this capacitance, called *distributed capacitance*, we find that the coil behaves, for practical purposes, as if in place of it we had an inductor $L$, resistor $R$, and capacitor $C$ connected as in Figure 74. In this figure two branches meet at the points $P_1$ and $P_2$, and are said to be in *parallel*. The capacitor $C$ is thus in parallel with the inductor and resistor.

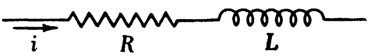

Fig. 73. Symbolic form of a coil, neglecting capacitance effects.

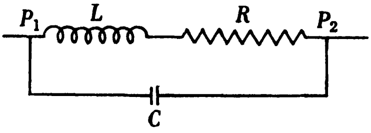

Fig. 74. Actual coil circuit.

3.18. **Kirchhoff's laws.** The following laws follow immediately from the definitions of current and difference of potential.

**Kirchhoff's first law.** *The algebraic sum of the potential drops around every closed circuit is equal to zero.*

**Kirchhoff's second law.** *The algebraic sum of the currents at any junction of conductors is zero.*

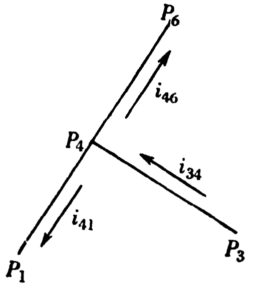

Fig. 75. Currents at a junction.

As an illustration of the first law consider the boundary of the network in Figure 71. Let $V_{ab}$ be the potential at the point $P_a$ minus the potential at $P_b$. By the first law we have for the boundary circuit of the network:

$$V_{12} + V_{27} + V_{76} + V_{65} + V_{51} = 0. \tag{3.75}$$

Consider the junction, or node, $P_4$ of Figure 71. We let $i_{ab}$ designate the current flowing from $P_a$ to $P_b$ along the branch $P_aP_b$. By Kirchhoff's second law we have:

$$i_{46} + i_{41} + i_{43} = 0, \tag{3.76}$$

or what is the same thing (see Figure 75)

$$i_{46} + i_{41} = i_{34}. \tag{3.77}$$

In an example where the currents are unknown we may imagine the currents as flowing in any assigned direction. The signs of the currents in the solution will give the correct picture. Thus if $i_{34}$ in Figure 75 turns out to be negative, the current is actually flowing from $P_4$ to $P_3$.

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# CHAPTER

# 4

# *Electric systems*

**4.1. Example of a resistance network.** We consider the network of Figure 76 where the problem is to find the currents in the branches.

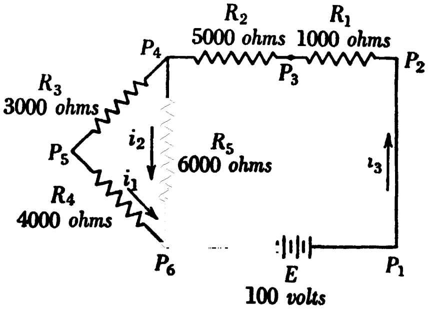

**Fig. 76. Resistance network.**

By Kirchhoff's first law the potential drop around the circuit $P_6P_5P_4P_6$ vanishes, whence

$$(R_3 + R_4)i_1 - R_5 i_2 = 0. \tag{4.1}$$

For the circuit $P_6P_1P_2P_4P_6$ we have

$$E - (R_1 + R_2)i_3 - R_5 i_2 = 0. \tag{4.2}$$

By Kirchhoff's second law we have

$$i_1 + i_2 = i_3. \tag{4.3}$$

Substituting the values of Figure 76 we obtain from (4.1), (4.2), and (4.3) the linear equations:

$$\begin{aligned} 7{,}000 i_1 - 6{,}000 i_2 &= 0, \\ 6{,}000 i_3 + 6{,}000 i_2 &= 100, \\ i_1 + i_2 &= i_3 \end{aligned} \tag{4.4}$$

Solution of (4.4) yields

$$i_1 = \frac{1}{200}, \quad i_2 = \frac{7}{1200}, \quad i_3 = \frac{13}{1200}. \tag{4.5}$$

The values of (4.5) are in amperes. Since all of these values are positive, the directions of the currents in Figure 76 are correct. Had we taken the direction of $i_1$ opposite to that of Figure 76 we would have had $-i_1$ in place of $i_1$ in the equations above, and the solution for $i_1$ would have been $-\frac{1}{200}$ ampere, that is, $-5$ (a milliampere is 0.001 ampere) milliamperes. We would then know that the current flowed in the opposite direction.

**4.2. Elements in series.** It is convenient to have a rapid method of handling like elements, such as capacitors, in parallel or in series.

That resistances and inductances are respectively additive when the corresponding resistors and inductors are in series follows immediately from the fact that the same current flows in elements in series. Thus the inductances of two inductors in series are mathematically equivalent to a single inductance equal to the sum of the given inductances. For this reason all of the inductances in one branch are lumped into one. Other like elements are also lumped.

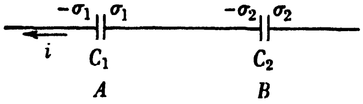

**Fig. 77. Capacitors in series.**

We shall consider capacitors $A$ and $B$ in series as in Figure 77. The horizontal line segments indicate conductors. We let $i$ designate the current in the left branch, and for simplicity shall suppose that the current of positive charges is flowing to the left in this branch as shown. The charge $-\sigma_1$ on the left plate of the capacitor $A$ is thus becoming more negative. We also assume for simplicity that $\sigma_1$ is positive. As the charge $-\sigma_1$ on the left plate decreases, at the same time, to balance, the charge $\sigma_1$ on the right plate increases at the same numerical rate. Since $i$ is the rate $dq/dt$ at which charge is flowing at any point on the left conductor, and the same must hold at the end point on the left capacitor plate, it follows that

$$i = \frac{d\sigma_1}{dt}. \tag{4.6}$$

Let us suppose that the conductor element between the capacitors was at some time in electric equilibrium with no net movement of charges in any direction. We let $\sigma_2$ denote the charge on the right plate of $B$. Corresponding to the charge $\sigma_1$ at one end of the middle conductor there must then be an equal but opposite charge $-\sigma_1$ at the other end, namely the left plate of capacitor $B$, whence

$$\sigma_1 = \sigma_2. \tag{4.7}$$

The current at each point on this middle conductor is the same and is therefore equal to the current at the left end. It follows that there is a current $i$ in the middle conductor where $i$ satisfies (4.6). Similarly, the current in the right branch is also $i$.

Capacitors may be suddenly inserted in a circuit so that $\sigma_1$ and $\sigma_2$ are not equal. It still follows that (4.6) is valid. The rate at which charge is changing on the left plate of $A$ is the negative of the rate for the right plate of this capacitor. Since the current everywhere is the same we have

$$\frac{d\sigma_1}{dt} = \frac{d\sigma_2}{dt}. \tag{4.8}$$

We let $\sigma_1(0)$ denote the charge on the right plate of the capacitor $A$ at a time $t = 0$, and $\sigma_2(0)$ the corresponding charge on the right plate of the capacitor $B$. We let $q(t)$ be such that the charge on the right plate of the capacitor $A$ is $\sigma_1(0) + q(t)$. The corresponding charge for the capacitor $B$ is $\sigma_2(0) + q(t)$. Then

$$i = \frac{dq}{dt}. \tag{4.9}$$

Let $C_1$ and $C_2$ be the capacitances of $A$ and $B$ respectively. The potential difference $e_1$ across the capacitor $A$ is now

$$e_1 = S_1[\sigma_1(0) + q(t)], \tag{4.10}$$

where $S_1$ is the *elastance* $1/C_1$. Differentiating (4.10) we have the simpler relation

$$\frac{de_1}{dt} = S_1 i = \frac{i}{C_1}. \tag{4.11}$$

Similarly, the potential difference $e_2$ across $B$ is such that

$$\frac{de_2}{dt} = S_2 i = \frac{i}{C_2}. \tag{4.12}$$

The potential difference across both capacitors is thus $e$ where

$$\frac{de}{dt} = \frac{de_1}{dt} + \frac{de_2}{dt} = (S_1 + S_2)i = \left(\frac{1}{C_1} + \frac{1}{C_2}\right) i. \tag{4.13}$$

Thus the *elastance of two capacitors in series is the sum of the elastances of these capacitors*. It follows that the *capacitance $C$, equivalent to the capacitances $C_1$ and $C_2$ of capacitors $A$ and $B$ in series, is such that*

$$C = \frac{C_1C_2}{C_1 + C_2}. \tag{4.14}$$

**4.3. Capacitors in parallel.** Any like elements, such as two resistors, in parallel are equivalent to a single element of the same kind.

We shall illustrate this with the capacitors of Figure 78 with capacitances $C_1$ and $C_2$. Here $i_1$ is the current in the $C_1$ branch, while $i_2$ is the current in the $C_2$ branch, and $i$ is the total current.

By Kirchhoff's first law the potential differences across the capacitors are equal, taken, of course, in the same direction. Hence by (4.11)

$$i_2 = \frac{C_2 i_1}{C_1}. \tag{4.15}$$

Since $i$ is the sum $(i_1 + i_2)$, we have

$$i_1 = \frac{C_1 i}{C_1 + C_2}. \tag{4.16}$$

Let $e_1$ be the potential difference across the capacitors. It follows that

$$\frac{de_1}{dt} = \frac{i}{C_1 + C_2}, \tag{4.17}$$

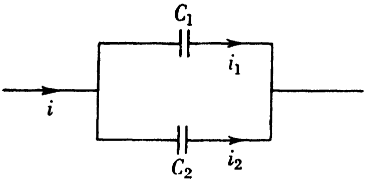

**Fig. 78. Capacitors in parallel.**

whence *the capacitances of two capacitors in parallel are equivalent to one capacitance equal to the sum of the given capacitances.* Thus by joining capacitors in parallel a capacitor with larger capacitance is obtained.

The same type of argument shows that the resistances $R_1$ and $R_2$ of resistors in parallel are equivalent to one resistance with the value $R_1R_2/(R_1 + R_2)$. Similarly, inductances $L_1$ and $L_2$ of inductors in parallel are equivalent to a single inductance with the value $L_1L_2/(L_1 + L_2)$.

Reluctances are treated like resistances as far as combining magnetic elements in parallel and in series is concerned.

**4.4. Capacitor discharging through a resistor.** We shall treat the example of Figure 79 where capacitors are both in series and parallel. The initial voltages across the capacitors are shown.

The capacitors in parallel in Figure 79 are equivalent to a third capacitor with the capacitance 0.3 μfd, which is the sum of the capacitances of the given capacitors. This third capacitor and the 0.3 μfd capacitor of Figure 79 are then in series. By (4.14) these two capacitors are equivalent to the capacitor of Figure 80. In this figure $e$ and $i$ denote instantaneous values of the voltage and current as shown. With the initial value of 110 volts for $e$, the instantaneous value of $e$ is given by

$$e = 110 + 67 \cdot 10^5 \int_0^t i\,dt. \tag{4.18}$$

From the resistor

$$e = -100{,}000i. \tag{4.19}$$

By (4.18) and (4.19)

$$10^5 i + 67 \cdot 10^5 \int_0^t i\,dt + 110 = 0, \tag{4.20}$$

which by differentiation yields

$$10^5 i' + 67 \cdot 10^5 i = 0. \tag{4.21}$$

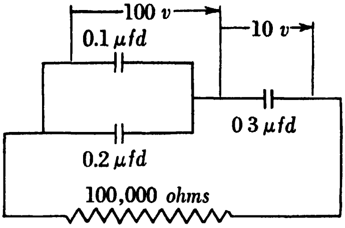

**Fig. 79. Capacitors discharging through a resistor.**

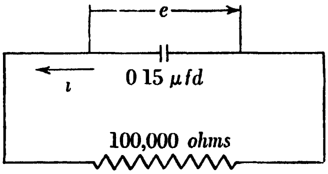

**Fig. 80. Circuit equivalent to that of Fig. 79.**

The solution of (4.21) is

$$i = Ce^{-67t}. \tag{4.22}$$

By (4.20), when $t = 0$ we have $i = -11 \cdot 10^{-4}$ ampere, or $-1.1$ milliamperes. The minus sign shows that current flows opposite to that shown in Figure 80. From this initial value of $i$ and (4.22) we have

$$i = -11 \cdot 10^{-4} e^{-67t}. \tag{4.23}$$

Substituting $i$ from (4.23) in (4.19) we obtain the formula for $e$ in terms of time, whence the problem has been completely analyzed.

The technique of differentiating the equation (4.20) to remove the integral is standard. Such integrals arise in the study of networks because of the capacitances in the network, and when the equations are expressed in terms of voltages instead of currents these integrals will come from inductances. Note that the initial condition was obtained from (4.20).

## 4.5. Coil in a circuit with a switch.

We suppose that a coil with inductance of 0.1 henry and resistance of 1,000 ohms is in series with a 10-volt battery as shown in Figure 81. The circuit is initially open with the switch in the position 1, and then the circuit is suddenly closed by moving the switch to the position 2.

The battery has internal resistance which we shall either neglect or assume has been included in the 1,000 ohms. In a typical case of a storage battery this resistance would be 0.001 ohm.

The potential difference across the inductance and resistance is $(0.1i' + 1{,}000i)$ for the instantaneous current $i$ and the time derivative $i'$. When the switch is closed this is 10 volts, whence we have

$$0.1i' + 1{,}000i = 10. \tag{4.24}$$

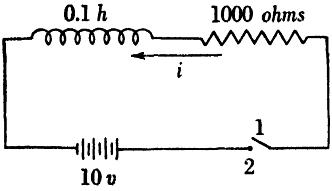

**Fig. 81. Coil in a circuit with a switch.**

Our initial condition is

$$t = 0,\ i = 0. \tag{4.25}$$

The relations (4.24) and (4.25) completely determine the solution for $i$.

**4.6. Circuit with switch opened.** Let us suppose that in the example of Figure 81 the switch is closed, and then is suddenly opened to the position 1. In the open position 1 the air between the points 1 and 2

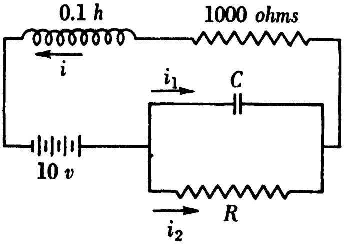

**Fig. 82. Circuit equivalent to that of Figure 81.**

of the figure acts like the dielectric of a capacitor of very small capacitance $C$ if the points are far apart. We shall neglect arcing effects. The air between the points 1 and 2 also has a resistance $R$. Normally $R$ is very large.

We replace the opened switch in Figure 81 by an equivalent capacitor and resistor as shown in Figure 82. We introduce the currents $i_1$ and $i_2$ as shown where

$$i = i_1 + i_2. \tag{4.26}$$

Considering the circuit without $C$ we have, by Kirchhoff's laws

$$0.1i' + 1{,}000i + Ri_2 = 10. \tag{4.27}$$

Since the drop across $C$ is the same as the drop across $R$, we have the additional relation

$$Ri_2 = E_0 + \frac{1}{C}\int_0 i_1 dt \tag{4.28}$$

for the initial voltage $E_0$ across the capacitor. At the start $i' = 0$ (just before the disturbance), whence by (4.24)

$$t = 0,\ i = 10, \tag{4.29}$$

where $i$ is in milliamperes. We may suppose that $C$ is uncharged at the start, so that

$$t = 0,\ E_0 = 0. \tag{4.30}$$

By (4.26), (4.28), and (4.30) we have

$$RCi_2' + i_2 = i. \tag{4.31}$$

Equations (4.27) and (4.31) may now be combined to yield a second-order equation in $i$. To determine $i$ the initial condition of (4.29) does not suffice. With

$$t = 0,\ i_2 = 0,$$

(4.27), and (4.29) we obtain the initial value of $i'$.

## 4.7. Alternating current generator in a circuit.

We shall treat the circuit of Figure 83 with a resistance of 20,000 ohms, capacitance of 1 microfarad and inductance of 100 henries. By convention an *alternating voltage* $V$ is given by $V\sqrt{2}\sin(\omega t + \phi)$ for constants $\omega$ and $\phi$, and time $t$ (see reference 7 at the end of this chapter). Alternating current is similarly defined. There is an alternating 110-volt 60-cycle source, so that the voltage $E$ across the generator may be given by

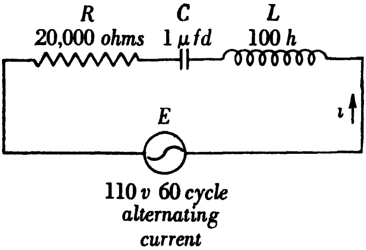

Fig. 83. Generator in a circuit.

$$E = 110\sqrt{2}\sin 120\pi t. \tag{4.32}$$

We introduce the current $i$ as shown. We have then

$$100i' + 20{,}000i + 10^6\int_0^t i\,dt = 110\sqrt{2}\sin 120\pi t. \tag{4.33}$$

We have neglected the initial charge of the circuit because we are clearly interested only in steady-state performance, and we shall differentiate (4.33) to obtain the equation to be solved for the steady-state component. As a matter of fact, after a sufficiently long time the state of the system will be independent of starting charge, so that this may be disregarded.

Differentiating (4.33) we have the equation

$$100i'' + 20{,}000i' + 10^6 i = 13{,}200\pi\sqrt{2}\cos 120\pi t. \tag{4.34}$$

Equation (4.34) can be solved for a unique steady-state solution $i$ by the theory of differential equations.

**4.8. Impedance.** Because it is convenient to have linear equations in place of differential equations it is customary in electrical engineering to use impedance and reactance.

The quantity $\sin 120\pi t$ of (4.33) is the imaginary part of the power $e^{120\pi jt}$. If we substitute this power for the term $\sin 120\pi t$ in (4.33), and solve for $i$ of the form $Ie^{120\pi jt}$, the imaginary part of this solution is the desired one. We shall give a general argument to justify this. Consider the differential equation

$$O(D)v = E_0 \sin \omega t, \tag{4.35}$$

where $O(D)$ is a polynomial in the derivative operator $D$ with real coefficients. By the theory of differential equations the equation (4.35) has the steady state solution $v$, where

$$v = A \sin \omega t + B \cos \omega t \tag{4.36}$$

for numbers $A$ and $B$. Differentiating both sides of (4.35) with $v$ as in (4.36), and dividing by $\omega$, we obtain

$$O(D)u = E_0 \cdot \cos \omega t, \tag{4.37}$$

where

$$u = A \cos \omega t - B \sin \omega t.$$

Combining (4.35) and (4.37), making use of

$$e^{j\omega t} = \cos \omega t + j \sin \omega t,$$

we obtain

$$O(D)(u + jv) = E_0 e^{j\omega t}. \tag{4.38}$$

Thus the steady state solution $v$ of (4.35) satisfies (4.38), where $u = v'/\omega$. Conversely, equation (4.35) can be obtained from (4.38) by equating imaginary parts, so that a solution $v$ of (4.38) also satisfies (4.35). Thus the steady state solution of (4.35) is the imaginary part of the steady state solution $u + jv$ of (4.38), which steady state solution is of the form

$$u + jv = I_0 e^{j\omega t},$$

where $I_0$ is $A + Bj$.

If in place of the right member of (4.35) we have $O_1(D)$ for a polynomial operator $O_1(D)$ with real coefficients, the treatment of (4.35) still applies, and the solution $v$ of

$$O(D)v = O_1(D) \sin \omega t$$

is the imaginary part of the solution of

$$O(D)(u + jv) = O_1(D)e^{j\omega t}.$$

Consider Figure 83 where we have denoted the inductance, resistance, and capacitance by $L$, $R$, and $C$ respectively. Let the given alternating voltage be written as

$$e = E_0 \sin \omega t.$$

The complex form of $e$ is then $E_0e^{j\omega t}$. The differential equation of the circuit

$$Li' + Ri + \frac{1}{C}\int_0^t i dt = E_0 \sin \omega t$$

is of the type (4.35), and corresponds to the complex form

$$\left(Lj\omega + R - \frac{j}{\omega C}\right)I_0 = E_0$$

obtained from (4.38) by setting $u + jv = I_0e^{j\omega t}$. This equation has the solution

$$I_0 = \frac{E_0}{R + j\left(L\omega - \frac{1}{\omega C}\right)}. \tag{4.39}$$

The advantage of writing $I_0$ as in (4.39) is that we can immediately obtain from it the amplitude and phase of $i$. In particular, the amplitude of $i$ is $\sqrt{A^2 + B^2}$, when $i$ is written in the same form as $v$ in (4.36), whereas the magnitude of $I_0$ is also $\sqrt{A^2 + B^2}$. From (4.39) we have

$$| i | = \frac{E_0}{\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}} \tag{4.40}$$

for the amplitude $| i |$ of $i$. We write

$$X_L = \omega L, \quad X_C = -\frac{1}{\omega C}, \quad Z = R + j(X_L + X_C). \tag{4.41}$$

The quantity $X_L$ is termed the *inductive reactance* of the circuit, $X_C$ the *capacitive reactance*, and $Z$ the *impedance*. Relation (4.38) for the case under consideration becomes

$$ZI_0 = E_0 \tag{4.42}$$

identical in form with Ohm's law.

These considerations generalize to any linear network with alternating voltage. By *linear network* we mean a network made up of resistances, inductances, capacitances, and potential sources, where the resistances, inductances, and capacitances satisfy the properties given for them in sections above. We shall illustrate the technique with a more complicated example.

Current that flows in one direction only is called *direct current.*

### 4.9. Impedance method of treating a network.

We shall derive complex algebraic linear equations for the network of Figure 84. We introduce the currents $i$, $i_1$, $i_2$ as shown so that $i$ is the sum of $i_1$ and $i_2$. We write

$$E = E_0 e^{j\omega t}, \quad i = Ie^{j\omega t}, \quad i_1 = I_1 e^{j\omega t}, \quad i_2 = I_2 e^{j\omega t}. \tag{4.43}$$

We shall leave it to the reader to verify that Kirchhoff's laws hold when applied to such quantities as $E_0$, $I$, $I_1$, and $I_2$.

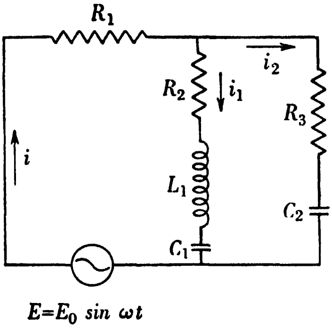

**Fig. 84. Alternating current network.**

From the left loop we have

$$R_1 I + Z_1 I_1 = E_0, \tag{4.44}$$

where

$$Z_1 = R_2 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right).$$

Going around the outer circuit we have

$$R_1 I + Z_2 I_2 = E_0, \tag{4.45}$$

$$Z_2 = R_3 - \frac{j}{\omega C_2}.$$

We also have

$$I_1 + I_2 = I. \tag{4.46}$$

The equations (4.44) – (4.46) yield the $I$'s in terms of $E_0$.

It follows readily from the theory of linear equations that if there are several sources of electromotive force at different frequencies, the current distribution can be obtained for each individual source, and the results added. Thus the principle of superposition holds for currents in a linear network with several sources of electromotive force. Through Fourier analysis the output of one generator may be considered to be due to several sources of electromotive force in series if the generator voltage wave is periodic but not sinusoidal.

**4.10. Impedance in a mechanical system.** Consider the spring-dashpot-mass system of Figure 16 treated in Section 2.5, but introduce a forcing function $B \sin \omega t$ so that the differential equation of motion becomes

$$mx'' + cx' + kx = B \sin \omega t \tag{4.47}$$

for constants $B$ and $\omega$. The right side of (4.47) is the imaginary part of the term $Be^{j\omega t}$, whence we consider the equation

$$mx'' + cx' + kx = Be^{j\omega t} \tag{4.48}$$

instead. Setting $x = Xe^{j\omega t}$ for a number $X$, equation (4.48) becomes

$$(-m\omega^2 + cj\omega + k)X = B. \tag{4.49}$$

The coefficient of $X$ may be thought of as the *mechanical impedance* of the system being studied. Because (4.47) is second-order, this is not precisely the analogue of electric impedance. By first integrating (4.47) and replacing the right side of the new equation by $B \sin \omega t$ we obtain an exact analogue.

These results can be extended to rotations of rigid bodies about axes.

**4.11. Mathematical formulation of a linear network.** We shall now set up the equations and initial conditions for the network of Figure 85 where the switch $S$ is first in the position 1 while the network is in a state of equilibrium, and then the switch is suddenly moved to the position 2, cutting out the battery. There is only one voltage source $E$ in the circuit for the position 1. The three horizontal stripes beneath the battery indicate *ground*, namely a convenient reference potential. The resistance $R_{L1}$ is the resistance of the coil with self-inductance $L_1$ forming the primary of an iron-cored transformer. The fact that the transformer is iron cored is indicated by the vertical lines between the primary with inductance $L_1$ and the secondary with inductance $L_2$. The mutual inductance between these coils is $M_{12}$. There is a second transformer in the circuit with primary of self-inductance $L_4$, secondary with self-inductance $L_5$, and mutual inductance $M_{45}$. This transformer is air cored. The resistance of the coil with inductance $L_2$ is $R_{L2}$, and similarly for the other coils.

We introduce the $i$'s as shown for the currents in the various branches. We shall suppose that the transformer secondaries are so connected that the electromotive forces due to mutual induction subtract as in the analysis to follow. From Kirchhoff's first law and the first circuit (most to the left in Figure 85) we have

$$(R_1 + R_{L1})i_1 + L_1 \frac{di_1}{dt} - M_{12}\frac{di_2}{dt} + \frac{1}{C_1}\left[q_{01} + \int_0^t i_1 dt\right] = 0, \tag{4.50}$$

when the switch is in the position 2, the quantity $q_{01}$ being the initial charge on the capacitor with capacitance $C_1$. Similarly, from other circuits

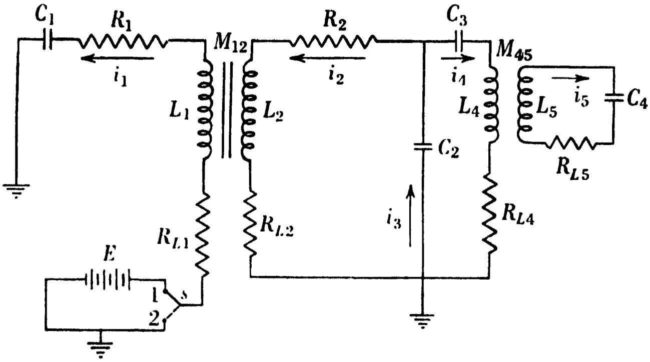

Fig. 85. Network with various elements.

we have

$$\begin{gathered}
\frac{1}{C_2}\left[q_{02} + \int_0^t i_3 dt\right] + (R_2 + R_{L2})i_2 + L_2\frac{di_2}{dt} - M_{12}\frac{di_1}{dt} = 0, \\
\frac{1}{C_2}\left[q_{02} + \int_0^t i_3 dt\right] + \frac{1}{C_3}\left[q_{03} + \int_0^t i_4 dt\right] + R_{L4}i_4 + L_4\frac{di_4}{dt} - M_{45}\frac{di_5}{dt} = 0, \\
R_{L5}i_5 + \frac{1}{C_4}\left[q_{04} + \int_0^t i_5 dt\right] + L_5\frac{di_5}{dt} - M_{45}\frac{di_4}{dt} = 0.
\end{gathered} \tag{4.51}$$

By Kirchhoff's second law we have

$$i_2 + i_4 - i_3 = 0. \tag{4.52}$$

Differentiation of equations (4.50) and (4.51) yields a system of second-order differential equations, which can be solved by the usual methods of ordinary differential equations if the initial values of the $i$'s and their derivatives are known.

Before the switch is moved from 1 to 2, the left circuit being in equilibrium, the current $i_1$ must be zero. If the current were never zero, the charge on the capacitor would build up forever, which would be impossible. This fact can also be seen from the differential equations of the network valid for the initial position of the switch.

It follows that for the switch position 1 the potential drop across $C_1$ equals the drop $E$ across the battery, whence

$$\frac{q_{01}}{C_1} = E. \tag{4.53}$$

Similarly,

$$q_{02} = q_{03} = q_{04} = 0, \tag{4.54}$$

and

$$i_1 = i_2 = i_3 = i_4 = i_5 = 0. \tag{4.55}$$

With the initial values of the $q$'s and $i$'s known, we can solve equations (4.50), (4.51), and (4.52) for the initial values of the derivatives of the $i$'s with respect to $t$. The problem is thus mathematically determined.

In treating the network of Figure 85 we have followed the usual technique of neglecting distributed capacitance effects in the transformers, variations in the inductors, and similar aspects normally of secondary importance.

**4.12. Electronic amplification stage.** We shall treat the standard amplification stage shown in Figure 86 with a triode tube. See Figure 87 for commercial triodes. The element 2 is the *cathode* heated by a filament from a voltage source not shown, so that the cathode emits electrons due to the thermal agitation of its atoms. The element 3 is the *anode*, or *plate*, which is relatively cold, and to which the electrons are attracted by the positive potential on it, where this potential is due to the voltage source $E$. The quantity $E$ need not be the constant potential difference across a battery, but in actual practice may be the potential difference across a power supply which rectifies given *alternating current*, to give *direct current. It is convenient to indicate a constant source of potential difference by a battery.*

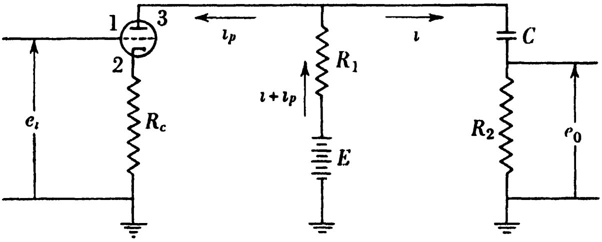

Fig. 86. Electronic amplification stage.

The plate being cold we would expect *no electronic current* from the plate to the cathode, and that is what happens, so that the tube is really a *rectifier tube*, for practical purposes passing current only one way even if the cathode is put at a higher potential relative to the plate.

The element 1 is the *grid*, which in the triode is placed between the cathode and plate. By varying the potential of the grid relative to the cathode the current between the cathode and plate can be correspondingly changed.

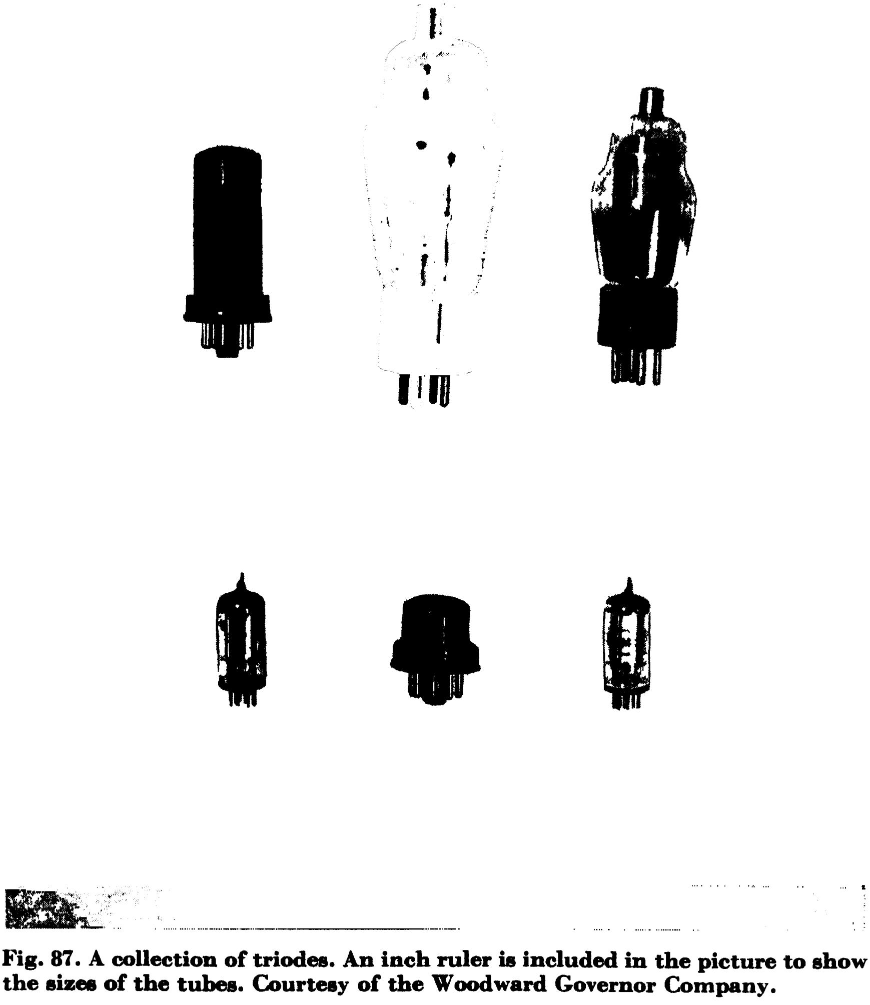

**Fig. 87. A collection of triodes. An inch ruler is included in the picture to show the sizes of the tubes. Courtesy of the Woodward Governor Company.**

The relation between the plate voltage $e_p$ (relative to the cathode), the grid voltage $e_c$ (also relative to the cathode), and the plate current $i_p$ (from plate to cathode) is shown in Figure 88 for a typical tube. By working within a region $R$ as shown we may suppose that

$$\Delta e_p = r_p \Delta i_p - \mu \Delta e_c, \qquad (4.56)$$

where $\Delta e_p$, $\Delta i_p$, and $\Delta e_c$ are the deviations of the values of $e_p$, $i_p$, and $e_c$ from those at a fixed point $P_0$ in $R$, the constant $r_p$ is the *plate resistance* of the tube, and $\mu$ is the *amplification factor*.

Let us return to Figure 86. We shall suppose that $e_t$ is an alternating voltage, and we can therefore write $e_t$ as

$$e_t = A \sin \omega t \tag{4.57}$$

for constants $A$ and $\omega$. We shall suppose that $A$ is small enough and the grid bias $R_c$ large enough so that the fluctuations in grid voltage are within the range of the region $R$. Clearly,

$$e_c = e_t - R_c i_p \tag{4.58}$$

Our *problem* will be to determine the output $e_0$ of the amplification stage.

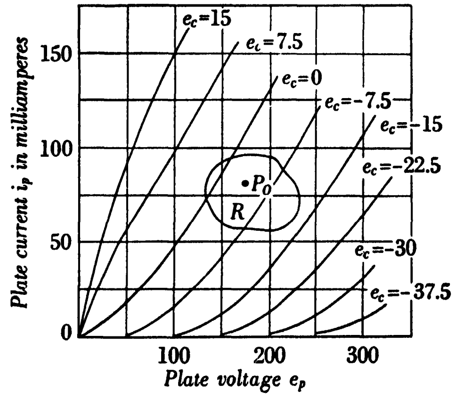

**Fig. 88. Characteristic curves of a triode.**

By Kirchhoff's first law, for the circuit with the elements $R_c$, the tube, $R_1$, and $E$ we have

$$E - R_1(i + i_p) - e_p - R_c i_p = 0. \tag{4.59}$$

In Figure 86 we have shown the current as going from the plate to the cathode in agreement with the convention of treating a flow of electrons in one direction as a flow of positive charges in the other.

By Kirchhoff's first law we have also the following for the right-hand circuit:

$$E - R_1(i + i_p) - \frac{1}{C}\left[q_0 + \int_0^t i dt\right] - R_2 i = 0, \tag{4.60}$$

where $q_0$ is the initial charge on the capacitor. Differentiating (4.60) we have the more convenient relation

$$R_1(i + i_p)' + \frac{i}{C} + R_2 i' = 0, \tag{4.61}$$

where the primes denote derivatives with respect to time. If we let $e_{p0}$, $i_{p0}$, and $e_{c0}$ be the values of $e_p$, $i_p$, and $e_c$ at the point $P_0$ we can write

$$e_p = e_{p0} + \Delta e_p, \quad i_p = i_{p0} + \Delta i_p, \quad e_c = e_{c0} + \Delta e_c. \tag{4.62}$$

We may choose $P_0$ so that we are at $P_0$ when $e_i \equiv 0$. Let the value of $i$ when we are at $P_0$ be designated by $i_0$ so that

$$i = i_0 + \Delta i. \tag{4.63}$$

By (4.61) we have $i_0 = 0$. The relations (4.58), (4.59), and (4.61) now yield

$$\begin{aligned} \Delta e_c &= e_i - R_c \Delta i_p, \\ R_1 \Delta i + \Delta e_p + (R_1 + R_c)\Delta i_p &= 0, \\ R_1(\Delta i + \Delta i_p)' + \frac{\Delta i}{C} + R_2(\Delta i)' &= 0. \end{aligned} \tag{4.64}$$

It is customary to use the symbols for the variables in place of the increments because we are interested only in deviations from equilibrium. With the driving function (4.57) we can solve (4.56) and (4.64) for the increments, and in particular for $\Delta i$, whence from

$$e_0 = R_2 \Delta i \tag{4.65}$$

we have the desired output. Since we are working with a steady state we do not need initial conditions, as the theory of ordinary differential equations shows.

For high frequencies such as $10^6$ cycles per second the interelectrode capacitances cannot be neglected, whence the network is more complicated.

If a nonsinusoidal periodic voltage $e_i$ is applied to the amplifier, the standard analytic technique is to break up this voltage into its *harmonic components by Fourier series*. From the principle of superposition the results for the given voltage can be obtained.

**4.13. Electric motor raising an elevator car.** We shall analyze an elevator where the car is raised by an electric motor as shown in Figure 89.

For the sake of simplicity we shall assume that the field of the electric motor is fixed and is stationary in the motor as indicated in Figure 90. We shall suppose that a constant potential difference $E$ is suddenly applied across the rotor of the motor as shown. The car is counterbalanced by a weight as in Figure 89 so that the motor must raise a net weight $w$, which we shall take in pounds. The car and counterweight are suspended from a pulley with radius $r$, whence there is a torque $T_w$ on the shaft of the motor given by

$$T_w = rw. \tag{4.66}$$

By Theorem 3.5 there is a torque $T_i$ due to the current $i$ in the rotor given by

$$T_i = ki \tag{4.67}$$

for a constant $k$. Letting $N$ be the angular velocity of the rotor in revolutions per minute (and hence of the shaft and pulley), and $I$ the moment of inertia of all of the moving parts relative to the axis of the shaft, the

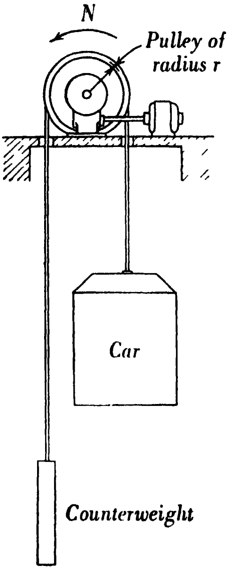

**Fig. 89. Elevator car raised by an electric motor.**

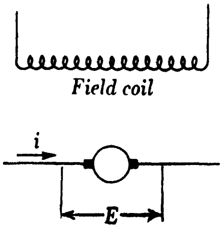

**Fig. 90. Electric motor with a fixed field.**

units being for example, in pounds, feet, and seconds, it follows from (2.55) that we have

$$\frac{\pi I N'}{30} + rw = ki. \tag{4.68}$$

Letting $R$ and $L$ be the resistance and inductance of the rotor we have

$$Li' + Ri = E. \tag{4.69}$$

As initial conditions we have

$$t = 0, \quad i = 0, \quad N = 0. \tag{4.70}$$

We are of course neglecting friction, windage, and other factors. These may be taken into account as in Part I.

In an elevator installation the field coil might be in series with the armature, or rotor, coil so that the same current $i$ flows through both, and the magnetic flux density $B$ is everywhere proportional to $i$. The proportionality factor will depend on the relative position of the rotor to the stator, but as a rough approximation we may take it constant, in which event the relation (4.67) is replaced by

$$T_i = ki^2 \tag{4.71}$$

and (4.68) is changed accordingly. A practical simplification of (4.69) is obtained by neglecting the $L$ term.

It is readily seen that *generator* problems are similar to motor problems.

**4.14. Transmission line.** We shall analyze a transmission line composed of two parallel conductors as shown in Figure 91. For a long line it is reasonable to suppose that the conductors will act like a capacitor so that there will be capacitance distributed along the line between the conductors. We would also expect current to leak from one conductor to the other through the air as the conducting medium. We shall take these factors into account, as well as the mutual inductance between the conductors and the self-inductance of each conductor.

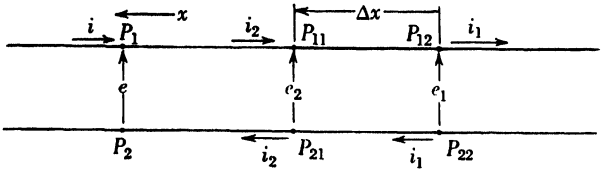

**Fig. 91. Transmission line composed of two parallel conductors.**

We consider a section of length $\Delta x$ of each conductor as shown in Figure 91, $x$ being the coordinate of length along the line. We let $i_2$ be the current through the point $P_{11}$ at the instant we are considering. By symmetry we have a current $i_2$ at $P_{21}$ directed as indicated. Similarly, we have currents equal to $i_1$ at the right ends of the elements as shown in the figure. We let $e_1$ and $e_2$ be the potential differences between corresponding points on the conductors as shown. We let $\frac{1}{2}R$ be the resistance of a conductor in ohms per unit length, assumed to be uniform, and $\frac{1}{2}L$ the inductance in henries per unit length, where the mutual and self-inductance are lumped. We let the distributed capacitance between the conductors be designated by $C$ farads per unit length. We let $R_1$ be the leakage resistance for a unit length.

The element of the top conductor in Figure 91 is equivalent to the branch in Figure 92 (a) composed of a resistance $\frac{1}{2}R\Delta x$ and inductance $\frac{1}{2}L\Delta x$ in series.

We let $e$ be the potential difference between two corresponding points $P_1$ and $P_2$ on the conductors as shown in Figure 91, whence $e_1$ and $e_2$ are particular values of $e$. In this figure $i$ is the current at $P_1$. Assuming that $e$

varies continuously along the transmission line, and possesses a continuous first derivative with respect to $x$ we can write:

$$e_2 = e_1 + \frac{\partial e}{\partial x}\Delta x + \epsilon_1, \tag{4.72}$$

where

$$\lim_{\Delta x \to 0} \frac{e_2 - e_1}{\dfrac{\partial e}{\partial x}\Delta x} = 1, \tag{4.73}$$

if

$$\frac{\partial e}{\partial x} \neq 0,$$

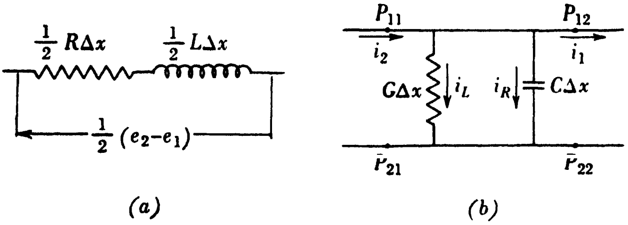

**Fig. 92. Circuits equivalent to an element of a transmission line.**

and the derivative in (4.73) is evaluated at $P_{12}$. By symmetry the potential at $P_{12}$ minus the potential at $P_{11}$ is the potential at $P_{21}$ minus that at $P_{22}$, whence the potential at $P_{11}$ minus that at $P_{12}$ is $\frac{1}{2}(e_2 - e_1)$. From Figure 92 (a) we have

$$e_2 - e_1 = (R\Delta x)i + L\Delta x\frac{\partial i}{\partial t}; \tag{4.74}$$

the current $i$ in the first term on the right is evaluated at some point on the element, and $\partial i/\partial t$ also at some point on the element. Relation (4.74) is set up on the assumption that $i$ and $\partial i/\partial t$ are continuous functions of $x$. From (4.72), (4.74), and the limit as $\Delta x$ goes to zero, we obtain

$$\frac{\partial e}{\partial x} = L\frac{\partial i}{\partial t} + Ri \tag{4.75}$$

valid at each section of the line. Typical values of $R$ and $L$ are 0.162 ohm and 0.002 henry per mile.

The quantity $G$, where $G = 1/R_1$, is termed the *leakage conductance.* We replace the section of the transmission line composed of the elements of the conductors of length $\Delta x$ by the network of Figure 92 (b), where the current $(i_2 - i_1)$ is split into two parts, one $i_L$ due to the conductance and the other $i_R$ due to the distributed capacitance. We assume that $\partial e/\partial t$ varies continuously along the transmission line. It follows that

$$i_L = (G\Delta x)e, \; i_R = (C\Delta x)\frac{\partial e}{\partial t}, \qquad (4.76)$$

where $e$ in the left equation of (4.76) is evaluated at some properly chosen section of the line, and the same is true of the derivative in the second equation. Since

$$i_2 - i_1 = i_L + i_R, \qquad (4.77)$$

and the analogue of (4.72) holds for $i$, the limit as $\Delta x$ goes to zero yields

$$\frac{\partial i}{\partial x} = C\frac{\partial e}{\partial t} + Ge. \qquad (4.78)$$

Equations (4.75) and (4.78) are the desired equations of the transmission line. For steady state alternating currents and voltages these equations may be solved by usual partial differential equation methods. For transients we must have initial conditions, such as the initial $e$ and $i$ at each section. A typical value of $C$ is $10^{-8}$ farads per mile. The quantity $G$ can generally be neglected.

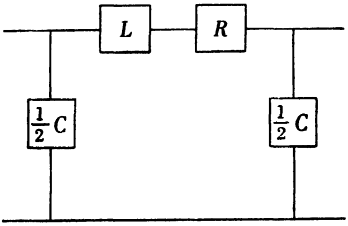

**Fig. 93. Block diagram of a transmission line with lumped elements.**

We can often lump the distributed electric elements for a transmission line, that is, replace the distributed self-inductance of the line by a single self-inductance, the distributed capacity by a single capacity, and the distributed resistances by a single resistance. When the electric elements are lumped, the partial differential equations of the system are replaced by ordinary differential equations, which are more easily treated. It is common practice to use 9 or 10 lumps per wavelength, each lump being composed of resistances, inductances, and capacitances. For 60 cycle alternating current the wavelength is the speed of light divided by 60, that is, approximately 3,100 miles. Thus for a 100 mile transmission line, and for this frequency, one lumping of the elements will do.

For a short power transmission line, not exceeding 50 miles in length, the capacitance of the line is often negligible, and its omission will generally introduce an error of not more than one per cent.

In Figure 93 we have a block diagram of a lumped transmission line for a typical three-phase line. The inductance and resistance of one component of the line, corresponding to one phase, are $L$ and $R$ respectively. The capacitance of the line relative to the ground is $C$.

**4.15. Ideal transformers.** We shall treat a transformer composed of primary and secondary coils wound on the same core. Let $N_1$ and $N_2$ denote the numbers of turns in these respective coils, let $i_1$ and $i_2$ be their respective currents, and let $v_1$ and $v_2$ be the respective voltages across the terminals of these coils.

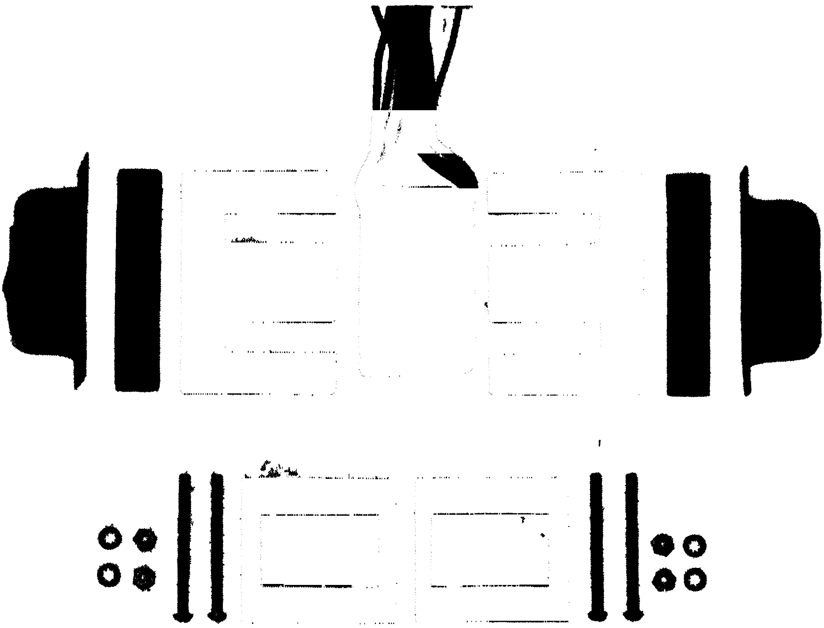

**Fig. 94. A transformer taken apart, showing the iron case (shield), the laminations forming the iron core, and the composite coil (center) formed from the primary and secondary wound for the same leg of the transformer. Courtesy of the Woodward Governor Company.**

We assume that we can neglect the resistances and capacitances of the windings, and that all the magnetic flux passes through each turn. We suppose that the permeability of the iron core is so high that a negligibly small magnetomotive force produces the required flux.

By Section 3.14 we have

$$v_1 = N_1 \frac{d\psi}{dt},$$

$$v_2 = N_2 \frac{d\psi}{dt}$$

for the flux $\psi$ crossing a surface bounded by one turn. Thus

$$\frac{v_1}{v_2} = \frac{N_1}{N_2}.$$

The magnetomotive force is assumed to be zero for practical purposes. If positive directions of the primary and secondary coil currents are taken properly we have

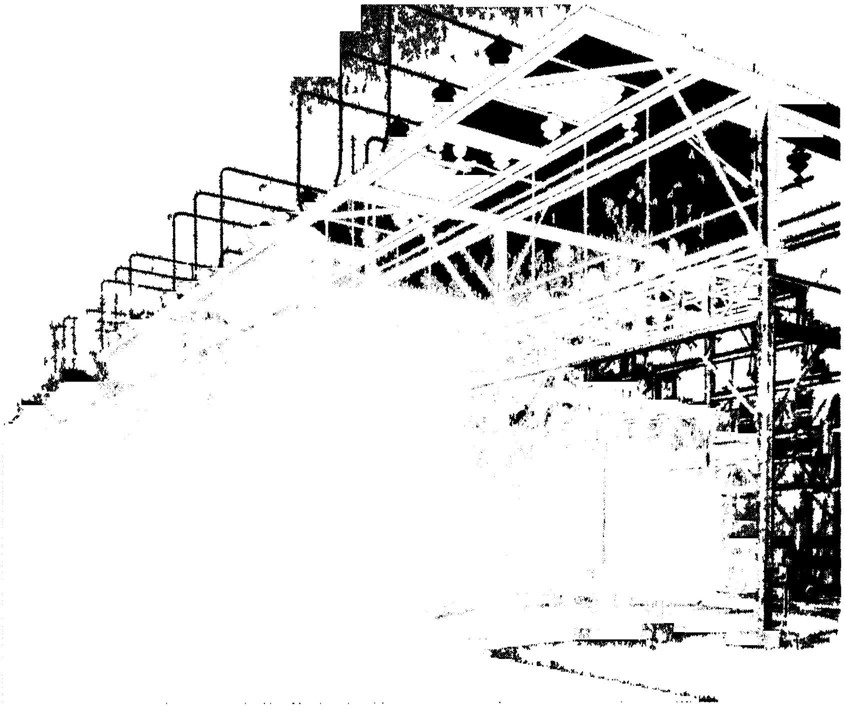

**Fig. 95. Substation with four power transformers. Courtesy of the General Electric Company.**

$$N_1 i_1 + N_2 i_2 = 0$$

so that

$$\frac{i_1}{i_2} = -\frac{N_2}{N_1}.$$

Practical experience shows that the simple expressions here obtained are adequate to explain transformer performance in many cases of varying currents.

For photographs of transformers see Figures 94 and 95.

**4.16. Permanent magnet direct current motor.** A permanent magnet direct current motor has a rotor (armature) which moves between the poles of a magnet. A coil is wound on the rotor. Let $\frac{1}{2}n$ be the number of turns in this coil, and $i$ the current. Actually, the rotor circuit may be composed of several coils in parallel, but for simplicity we shall refer to only one rotor coil. In what follows we may neglect the parts of the turns of the rotor coil not parallel to the rotor axis, and suppose that the rotor has $n$ conductors of length $l$ parallel to the rotor axis at a distance $r$ from this axis. We let $B$ denote an average flux density of the magnetic field so that the force on a conductor is $Bli$, and the corresponding torque on the rotor is $Blir$. Thus the total torque on the rotor is $nrBli$ (see Theorem 3.5).

Let $N$ be the speed of the rotor, and $I$ its moment of inertia. An equation of motion for the rotor is given by

$$I\frac{dN}{dt} + cN = nrBli \tag{4.79}$$

for time $t$, and a term $cN$ to take care of friction and windage. The term $cN$ may often be neglected.

We let $e$, $L$, and $R$ designate the voltage applied to the coil, the inductance, and resistance of the coil respectively. The symbol $e_c$ will denote the counter electromotive force generated in the coil due to the motion of the conductors in the magnetic field. By Theorem 3.6

$$e_c = nrlBN. \tag{4.80}$$

The current $i$ now satisfies the equation

$$L\frac{di}{dt} + Ri = e - e_c. \tag{4.81}$$

Equations (4.79)–(4.81) describe the motor.

Equations (4.79) and (4.80) are not in a convenient form for practical use. Suppose that $e$ and $e_c$ are in volts, $i$ is in amperes, $N$ in rpm, $B$ in lines per square inch, $t$ in seconds, $L$ in henries, $R$ in ohms, $I$ in inch-pound seconds squared, while $r$ and $l$ are in inches. Equations (4.79) and (4.80) become

$$\frac{\pi I}{30}\frac{dN}{dt} + c_1N = 885 \times 10^{-10}nrBli, \quad e_c = \frac{\pi}{3}10^{-9}nrlBN$$

respectively, where $c_1$ is a constant. Equation (4.81) still applies.

**4.17. Direct current series motor.** In a series motor the same current $i$ flows through the armature as through the stationary windings (*field*). The average flux density $B$ of the magnetic field is proportional to $i$, whence the torque on the rotor is $Ki^2$ for a constant $K$. The counter electromotive force $e_c$ is proportional to $BN$, where $N$ is the speed of the rotor, whence $e_c$ is $kiN$ for a constant $k$. The torque equation is given by

$$\frac{\pi I}{30}\frac{dN}{dt} + cN = Ki^2,$$

where $cN$ is the damping torque, and the first term is the inertial torque.

Let $e$ be the electromotive force applied to the motor. The current $i$ satisfies the equation

$$L\frac{di}{dt} + Ri = e - kiN.$$

**4.18. Induction motor.** The induction motor is one type of alternating current motor in common use. This motor is essentially a transformer with a moving secondary. The field (primary) is not electrically connected to the secondary circuit. A current is induced in the rotor by the varying magnetic field of the stator (stationary member). There is a synchronous speed $N_s$ corresponding to the rotating magnetic field of the stator. Let $N$ be the speed of the rotor. In practice $N_s > N$. If we had an ideal case, the same flux would pass through the stator as through the rotor, and the voltage $v_1$ supplying the field would satisfy the relation for $v_1$ in Section 4.15, where $v_1$ is proportional to $d\psi/dt$.

There is an electromotive force induced in the rotor due to the motion of the rotor, and one due to the variation of the field (that is, the voltage supplying the field) with respect to time. Suppose that the voltage supplied to the field is constant. When the rotor does not move relative to the rotating field, in which case $N = N_s$, there is no electromotive force generated in the rotor. We take the net secondary alternating voltage $v_2$ proportional to $N_s - N$ when $N_s - N$ is not too large. We suppose that the same is true of the secondary current $i_2$, whence the driving torque may be assumed to be proportional to $N_s - N$. We have the equation

$$I\frac{dN}{dt} = k(N_s - N)$$

for a constant $k$, time $t$, and moment of inertia $I$.

For a more accurate analysis of the electrical features of the motor, the resistances of the windings and the leakage flux must be taken into account. This can be done by replacing the stator and rotor circuits by equivalent circuits of inductances and resistances.

**4.19. Direct current generator.** Let $N$ be the speed of the generator, and $i_f$, $e_f$, $L_f$, and $R_f$ the field current, voltage, inductance, and resistance respectively, and let the corresponding symbols for the armature be the same except that the subscript $f$ is replaced by $a$. The field voltage is supplied by some external source, such as another generator called the *exciter*. Let $M$ denote the load. We have the equations

$$\begin{aligned} L_f \frac{di_f}{dt} + R_f i_f &= e_f, \\ M + L_a \frac{di_a}{dt} + R_a i_a &= e_a. \end{aligned} \tag{4.82}$$

In writing the first equation of (4.82) we have neglected the counter electromotive force due to the magnetic field of the armature current. This electromotive force can be taken care of, when necessary, by inserting a term proportional to $Ni_a$. We might also have a term in this equation proportional to $di_a/dt$.

It is common practice to have $R_f$ approximately equal to $L_f$, and $R_a$ about 10 times $L_a$.

The electromotive force $e_a$ is given by

$$e_a = kNi_f \tag{4.83}$$

for a constant $k$.

The torque equation is

$$I\frac{dN}{dt} + Ki_f i_a = T \tag{4.84}$$

for a constant $K$, moment of inertia $I$, time $t$, and driving torque $T$.

Equations (4.82)-(4.84) are four equations in the four dependent variables $i_f$, $i_a$, $e_a$, and $N$.

**4.20. Voice coil.** In Figure 96 we picture a cross section of a voice coil with the iron part in the form of the capital letter $E$. A practically uniform flux density is produced in the air gaps by a coil wound around the middle leg of the $E$, and indicated by cross hatching. Another coil fits into the air gaps free to move toward and away from the base of the $E$. This coil is spring loaded.

Voice coils are used in dynamic radio speakers. They can be made round so that there is but one air gap in the form of a circular ring.

The dashed line in the figure indicates a magnetic path. Let $\phi$ be the flux across the middle leg of the circuit, and $\mathcal{F}$ the magnetomotive force. If $\mathcal{R}_i$ and $\mathcal{R}_a$ designate the reluctances of the iron and air paths respectively, we have

$$\mathcal{F} = \mathcal{R}_a\phi + \mathcal{R}_i\phi.$$

The reluctances are known from the $B$-$H$ curves for the iron and air. If $i$ is the current in the fixed coil, and $N$ the number of turns, the force $\mathcal{F}$ is $Ni$. Thus if $i$ is given, we can find the flux density $B$ in the air gap, or gaps. We may take this flux density to be uniform.

The fixed coil may be omitted, and a permanent magnet used instead. In any case $B$ can be assumed known for what follows.

Let $i_c$ be the current in the movable coil. There is then a force on this coil in the direction of the axis of the coil, and proportional to $Bi_c$. Let $x$ be the axial coordinate of the coil. The coil now satisfies the equation

$$mx'' + cx' + kx = KBi_c,$$

where $m$ is the mass of the moving part, $c$ is a damping coefficient, $k$ a spring scale, and $K$ a constant of proportionality. The prime denotes differentiation with respect to time.

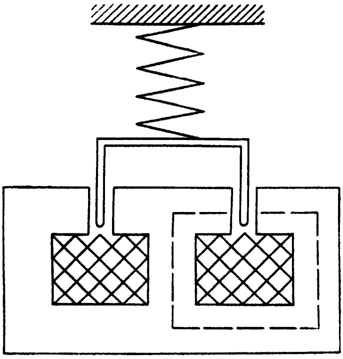

Fig. 96. Voice coil.

If a voltage $e_c$ is applied to the movable coil, the current $i_c$ satisfies

$$Li_c' + Ri_c = e_c \tag{4.85}$$

for the inductance $L$ and resistance $R$ of the movable coil.

**4.21. Microphone.** A common type of microphone is of the voice coil variety treated in the preceding section. We use the same notations as in that section except that $-K$ will be employed in place of $K$. The movable coil is attached to a diaphragm fixed at the edge of the diaphragm so that the diaphragm supplies a spring force. Let $G$ denote the force on the diaphragm due to the impingement of sound waves. The force equation for the diaphragm-coil is

$$mx'' + cx' + kx + KBi_c = G.$$

The induced electromotive force $e_c$ is given by

$$e_c = ABx' \tag{4.86}$$

for a constant $A$.

Equation (4.85) again applies except for external circuit terms. We thus have three equations in the three unknowns $x$, $i_c$, and $e_c$.

**4.22. Low pass filter.** Suppose that we have a resistor of resistance $R$ in series with a capacitor of capacitance $C$. The capacitor is in parallel with a load resistor of resistance $R_L$. The system described forms a *filter*. A voltage $e_i$ is supplied to the system. The output of the filter is the voltage $e_0$ across the load resistor. It is desired to obtain $e_0$ in terms of $e_i$. See Figure 97.

Let $i$ be the current through the capacitor, and $i_L$ the current through the load resistor. The voltage drops across the resistors yield

$$\begin{aligned} R(i + i_L) + e_0 &= e_i, \\ R_L i_L &= e_0. \end{aligned} \tag{4.87}$$

The voltage drop across the capacitor is $e_0$, so that

$$e_0 = \frac{1}{C}\int_0^t i\,dt + E_c \tag{4.88}$$

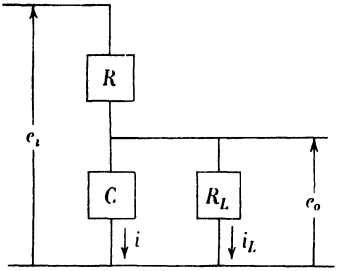

Fig. 97. Block diagram of a low pass filter.

for the initial voltage drop $E_c$ across the capacitor, and time $t$. By (4.88) the current $i$ is $Ce_0'$. By (4.87) we now have

$$RCe_0' + \left(1 + \frac{R}{R_L}\right)e_0 = e_i. \tag{4.89}$$

As usual, the primes denote derivatives with respect to time. The performance of the filter is determined by equation (4.89) and the initial condition $t = 0$, $e_0 = E_c$.

If we are interested in the response of the filter to sinusoidal inputs, we replace $e_0$ and $e_i$ in (4.89) by $E_0e^{j\omega t}$ and $E_ie^{j\omega t}$ respectively, with $j^2 = -1$. In this case (4.89) becomes

$$j\omega RCE_0 + \left(\frac{R + R_L}{R_L}\right)E_0 = E_i. \tag{4.90}$$

We can take $e_i$ and $e_0$ to be the imaginary parts of $E_ie^{j\omega t}$ and $E_0e^{j\omega t}$ respectively. Equation (4.90) is algebraic. By (4.90) the amplitude $E_0$ of $e_0$ satisfies

$$|E_0| = \frac{E_i}{\sqrt{\left(1 + \frac{R}{R_L}\right)^2 + (RC\omega)^2}}.$$

It is seen from this relation that, if $\omega$ is small enough, $| E_0 |$ is, for practical purposes, proportional to $E_i$. As the frequency increases beyond all bounds, the quantity $| E_0 |$ tends to zero. We thus have a low pass filter.

**4.23. Neon tube.** There is a large range of currents for which the voltage drop across a neon tube is practically independent of the current, for which reason the tube can be used for voltage regulation. We suppose that a neon tube $N$ is in series with a resistor of resistance $R$, and in parallel with a load resistor of resistance $R_L$. Let $i$ be the current through the tube, $i_L$ the current through the load resistor, $e_n$ the voltage drop across the tube, and $e$ the voltage drop across the system. See Figure 98. We have the equations

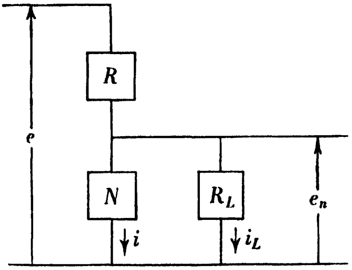

Fig. 98. Neon tube circuit.

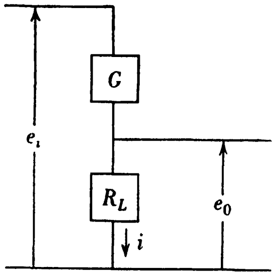

Fig. 99. Rectifier stage.

$$\begin{aligned} R_L i_L &= e_n, \\ R(i + i_L) + e_n &= e, \\ e_0 + \epsilon(i - i_0) &= e_n, \end{aligned}$$

where $i_0$ is a value of $i$ around which we are working, and $e_0$ is the value of $e_n$ which corresponds to $i_0$. The quantity $\epsilon$ is numerically small.

**4.24. Rectifier stage.** Suppose that we have a rectifier element $G$ in series with a load resistor of resistance $R_L$. See Figure 99. This rectifier element may be a selenium or germanium cell, an electronic diode, or some other rectifying device. An input voltage $e_i$ is applied to the system. We are interested in the output voltage $e_o$ across the load resistor.

We let $R$ designate the resistance of the rectifier element, and $i$ the current through this element. We may suppose that $R = R_0$ for $i \geqq 0$, and that $R$ is, for practical purposes, infinite if $i < 0$ (rigorously, the voltage drop across $R_L$ is zero if $i$ is not positive). Then

$$\begin{aligned} e_0 &= 0, \quad & e_i &< 0, \\ \left(1 + \frac{R_0}{R_L}\right) e_0 &= e_i, \quad & e_i &\geqq 0. \end{aligned}$$

**4.25. Transistor.** A *transistor* is a block of germanium or some other properly chosen material in contact with three electrodes as described below. See Figure 100. There is a large area low resistance contact $c_b$ on the base of the block. Two point contacts $c_e$ and $c_c$ on the upper surface of the block are the *emitter* and *collector* respectively. These two contacts are very close together, such as a distance of 0.002 to 0.01 inches. The transistor can be used as an amplifier. It has the advantage that it needs no warmup time, as is required in the case of electronic tubes.

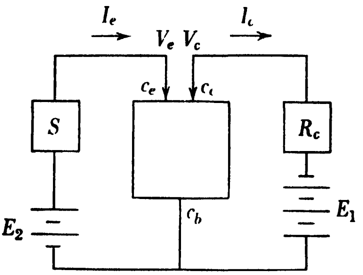

Fig. 100. Transistor.

The emitter is connected to the base in series with the signal element $S$ and a bias voltage, which places the emitter at a higher potential than the base when the signal element has zero potential drop. The collector is connected to the base in series with the load and a voltage which biases the collector negatively relative to the base. We let $R_c$ be a resistance representing the load, $V_e$ the emitter voltage relative to the base, $V_c$ the collector voltage relative to the base, $I_e$ the emitter current, and $I_c$ the collector current. Let $E$ be the signal voltage. Then

$$R_cI_c - V_c = E_1,$$
$$V_e - E = E_2,$$

for the bias voltages $E_1$ and $E_2$. Experiment shows that regardless of the bias

$$I_e = f(V_e + R_FI_c)$$

for a constant $R_F$ and a function $f$. An additional relation

$$g(V_e, V_c, I_e, I_c) = 0$$

is needed to complete a set of four equations in $V_c$, $V_e$, $I_c$, and $I_e$.

**4.26. Transductor.** Alternating voltage is applied to windings of two transformers, these windings being connected in series. We shall call these windings "ac windings." The other two windings of the transformers are connected in series, so that the alternating voltage induced in them cancels out. We shall call these windings the "dc windings." The configuration composed of the two transformers is a *transductor*. It may be used as an amplifier when the alternating current is rectified and used for feedback. Like the transistor it requires no warmup time.

As the current in the dc circuit of the transductor is varied the current in the ac windings is also changed.

The inductance, resistance, current, and voltage of the dc circuit will be designated by $L$, $R$, $i$, and $e$ respectively.

For the dc circuit we have the equation

$$Li' + Ri = e,$$

where the prime denotes differentiation with respect to $t$. An average value of $L$ should be used here. This value can be determined from a study of the flux in the transductor.

Let $I_{av}$ and $E_{av}$ respectively denote the average alternating current and voltage at the terminals of the ac windings of the transductor. Experiments show that for a large class of applications the relation between $E_{av}$, $I_{av}$, and $i$ can be expressed by characteristic curves so that there is a function $f(E_{av}, I_{av}, i)$ for which

$$f(E_{av}, I_{av}, i) = 0.$$

The reader should be reminded of the characteristic curves of triodes (see Figure 88). Suppose that the load is a resistor in series with the ac windings, and that an alternating voltage with average value $E_0$ is supplied to the ac windings and load. Let $R$ denote the total resistance in the ac circuit. A study of average currents and voltages shows that there is a load line with the equation

$$E_{av} + RI_{av} = E_0.$$

We now have three equations in the variables $I_{av}$, $E_{av}$, and $i$.

As an approximation to the performance of the transductor we can use

$$I_{av} = kI,$$

where $k$ is a constant of proportionality and $I$ is the deviation of $i$ from an equilibrium value.

Transductors may have the ac windings in parallel instead of series.

**4.27. Procedure for setting up electrical problems.** It is advisable, when possible to replace the electric elements by an equivalent network of resistances, inductances, capacitances, and sources of constant or variable current or voltage. If a conductor is not coiled, not exceptionally long, and the voltage and current changes in it are not rapid, we can usually neglect inductance and capacitance effects for the conductor. In the case of extensive neighboring conducting surfaces capacitance must generally be taken into account. It is usually a good approximation to assume that dielectric constants and magnetic permeabilities remain near their mean values. If a wide range of frequencies is being considered, such as $10^3$ to $10^6$ cycles per second, the variation of all electric elements may have to be studied. These variations must also be considered if wide temperature ranges occur, such as 0 degrees Fahrenheit to 500 degrees Fahrenheit. Pressure changes of 100 per cent or more may cause large changes in leakage conductance and other constants. If magnetic fields, as in electromagnets, are used, a sort of mean value of flux can often be assumed to hold over the whole space affected. The same is true of generators.

**4.28. Dimensional analysis applied to a transmission line problem.** Suppose that we have a long transmission line, and that a constant voltage $E$ is suddenly applied to the line at the input end. Until the first reflection from the load end of the line, the input current $i$ is independent of the load. We assume the line to be uniform with the resistance $R^*$, inductance $L^*$, and capacitance $C$ per unit length. We neglect leakage. It is desired to obtain the input current $i$ as a function of time $t$.

The secondary quantities in this problem are $E$, $i$, $R^*$, $L^*$, $C$, and $t$. We may take resistance, current, length, and time as primary quantities, denoted by the symbols $R$, $I$, $L$, and $T$ respectively. The dimensions of the secondary quantities follow. An explanation of these dimensions will be given immediately.

| QUANTITY | DIMENSIONS |
|---|---|
| $E$ | $RI$ |
| $i$ | $I$ |
| $R^*$ | $RL^{-1}$ |
| $L^*$ | $RTL^{-1}$ |
| $C$ | $R^{-1}TL^{-1}$ |
| $t$ | $T$ |

An inductance is a potential difference divided by the derivative of a current with respect to time. Inductance thus has the dimensionality $RI$ divided by $IT^{-1}$, or simply $RT$. Since $L^*$ is inductance per unit length we must divide by $L$ to obtain the dimensionality of $L^*$. Since capacitance is the quotient of a current by the derivative of a potential difference with respect to time, the dimensionality of a capacitance is $R^{-1}T$. The dimensionality of $C$ is an immediate consequence. The reader can readily verify the remaining entries in the table above.

A dimensionless product of the secondary quantities will be written as $E^a i^b R^{*c} L^{*d} C^e t^f$. Since that is dimensionless

$$\begin{array}{llll} a + & c + d - e & & = 0, \\ a + b & & & = 0, \\ & -c - d - e & & = 0, \\ & \phantom{-c -{}} d + e & + f & = 0. \end{array}$$

We solve this system of equations for the two cases where $b = 1$, and where $f = 1$. We thus obtain the dimensionless terms

$$\frac{R^*t}{L^*}, \frac{i}{E}\sqrt{\frac{L^*}{C}}.$$

It follows that

$$i = E\sqrt{\frac{C}{L^*}}f\left(\frac{R^*t}{L^*}\right)$$

for a function $f$.

We could have obtained the same results by using some other system of primary quantities (see the Appendix).

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# CHAPTER

# 5

# *Electromagnetic fields*

**5.1. Maxwell's equations for electromagnetic fields.** The material of this chapter is needed for the proof of the Stefan-Boltzmann radiation law of Chapter 8.

Consider a curve $C$ which is the boundary of a surface $\Sigma$, where a magnetic intensity $\boldsymbol{H}$ is defined at each point of $\Sigma$. Let $\boldsymbol{E}$ be the electric intensity at a point of $C$ associated with $\boldsymbol{H}$. Let $\psi$ be the flux across $\Sigma$. By Theorem 3.8

$$e = -\frac{d\psi}{dt}.$$

Since

$$e = \int_C \boldsymbol{E} \cdot d\boldsymbol{s},$$

and

$$\psi = \iint\limits_{\Sigma} \boldsymbol{B} \cdot d\boldsymbol{\Sigma},$$

for $\boldsymbol{B} = \mu \boldsymbol{H}$, we have

$$\int_C \boldsymbol{E} \cdot d\boldsymbol{s} = -\frac{d}{dt} \iint\limits_{\Sigma} \boldsymbol{B} \cdot d\boldsymbol{\Sigma} = -\iint\limits_{\Sigma} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{\Sigma},$$

assuming that the differentiation under the integral sign is allowed.

Assuming that we can apply Stokes's theorem of the calculus to the integral of $\boldsymbol{E} \cdot d\boldsymbol{s}$ around $C$ we have

$$\int_C \boldsymbol{E} \cdot d\boldsymbol{s} = \iint\limits_{\Sigma} (\mathbf{curl}\ \boldsymbol{E}) \cdot d\boldsymbol{\Sigma}.$$

Thus

$$\iint\limits_{\Sigma} (\mathbf{curl}\ \boldsymbol{E}) \cdot d\boldsymbol{\Sigma} = -\iint\limits_{\Sigma} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{\Sigma}.$$

Let us consider the equation

$$\iint\limits_{\Sigma} \left(\mathbf{curl}\ \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}\right) \cdot d\boldsymbol{\Sigma} = 0.$$

Assuming that this equation holds for a sufficiently large class of surfaces it follows that

$$\text{curl } E = -\frac{\partial B}{\partial t}, \tag{5.1}$$

or if we wish

$$\text{curl } E = -\mu \frac{\partial H}{\partial t}, \tag{5.2}$$

when $\mu$ is constant.

Equation (5.2) is a triple of Maxwell's equations written in vector form. This vector equation represents three equations relating the $x$-, $y$-, and $z$-components of the vectors involved.

Let us consider an infinitely long wire in the form of a straight line along which a constant current $i$ is flowing (see Figure 101). By the Biot-Savart theorem there is a contribution $dH$ to the magnetic intensity $H$ at a point $P$ due to an element of length $dl$ where

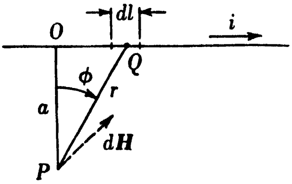

**Fig. 101. Straight conductor with a current $i$.**

$$| dH | = \frac{i(\sin \theta)dl}{r^2}.$$

Here $r$ is the distance from $P$ to some properly chosen point $Q$ on the element, and $\theta$ is an angle between the line from $Q$ to $P$ and the line of the wire. Let $O$ be the perpendicular projection of $P$ on the line of the wire and let $a$ be the distance $OP$. Let an angle from $PO$ to $PQ$ be denoted by $\phi$. The magnetic intensity $H$ at $P$ due to the current in the wire then has the magnitude $| H |$, where

$$| H | = i \int_K \frac{(\sin \theta)dl}{r^2} = i \int_K \frac{(\cos \phi)dl}{r^2} = \frac{i}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \phi d\phi = \frac{2i}{a},$$

and $K$ denotes the conductor. For Figure 101 the vector $H$ is perpendicular to the plane of the paper, and is pointed into the paper.

Let us evaluate the line integral of $H \cdot ds$ around a circle of radius $R$ with center on the wire, and in a plane perpendicular to the wire. The magnetic intensity is everywhere tangent to this circle. It follows that

$$\int_G H \cdot ds = \int_G | H |ds = \frac{2i}{R}\int_G ds = \frac{2i}{R}(2\pi R) = 4\pi i,$$

where $G$ is the circle. Thus the integral around the circle (in the direction of **H**) is independent of $R$.

Let $AB$ be an arc of a circle of radius $r_m$ with center on the wire and plane perpendicular to the wire, and let $\alpha$ be the smallest positive angle in radians subtending the arc $AB$ at the center of the circle. Let $CD$ be the arc of a circle of radius $r_M$ where the center of this circle is on the wire, the wire is perpendicular to the plane of the circle, this arc subtends the same central angle $\alpha$, and the line through $A$ and $D$ and the line through $C$ and $B$ pass through the wire. We consider the curve $ABCD$ formed from the arc $AB$, the straight-line segment $BC$, the arc $CD$, and the straight-line segment $DA$. Along the straight-line segments the vector $\boldsymbol{H}$ is perpendicular to these segments, whence the integral of $\boldsymbol{H} \cdot d\boldsymbol{s}$ along these segments vanishes. For the arc $CD$ we have (with $\mathbf{H} \cdot d\mathbf{s} = |\mathbf{H}|\, ds$)

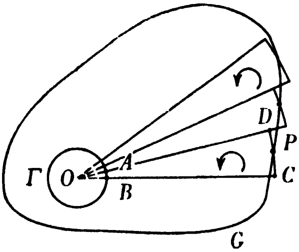

Fig. 102. Region approximated by sectors.

$$\int_C^D \boldsymbol{H} \cdot d\boldsymbol{s} = \frac{2i}{r_M}\int_C^D ds = \frac{2i}{r_M}(\alpha r_M) = 2i\alpha.$$

For the arc $AB$ we then obtain

$$\int_A^B \boldsymbol{H} \cdot d\boldsymbol{s} = -2i\alpha,$$

the minus sign being due to integrating in the opposite direction to the integral over $CD$. Thus the integral around the curve $ABCD$ vanishes. This curve does not enclose the wire.

The arcs $AB$ and $CD$ used above are not necessarily in the same plane.

Suppose now that we have an arbitrary curve $G$ around the conductor $K$ as in Figure 102. For simplicity we consider a curve $G$ in a plane perpendicular to $K$. We can enclose the wire with a circle $\Gamma$ in the plane of $G$ where the center $O$ of the circle is on the wire, and the circle lies entirely within the region bounded by the curve $G$. To simplify the argument we assume that a ray emanating from $O$ in the plane of $G$ cuts $G$ in one and only one point. The argument is readily extended to more general curves. We can take a point $P$ on $G$ and draw through this point an arc $CD$ of a circle with $O$ as center. Through the end points of this arc we can draw radii to $O$ cutting out an arc $AB$ on $\Gamma$. With the proper labeling of the points we thus obtain a region $ABCD$ as in Figure 103. We can obviously put in a finite number of such regions, so that these regions do not overlap, but each point of $\Gamma$ is on at least one of these regions. We can integrate $\boldsymbol{H} \cdot d\boldsymbol{s}$ in the positive direction around each region. The sum of these integrals for all of the regions is then zero. The integral of $\boldsymbol{H} \cdot d\boldsymbol{s}$ along the common boundary of two regions may be neglected because we traverse this boundary in opposite directions in integrating around these two regions, whence the sum of the integrals of $\boldsymbol{H} \cdot d\boldsymbol{s}$ along this common boundary for the two regions vanishes. The sum of the integrals around the regions is then the integral around $\Gamma$ in the negative direction with respect to the interior of the circle, plus the integral around the curve $G^*$ in the positive direction, where $G^*$ is the outer boundary of the region composed of the regions of type $ABCD$. It follows that

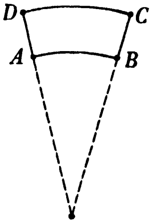

**Fig. 103. Region *ABCD* bounded by arcs of circles and straight line segments.**

$$\int_{G^*} \boldsymbol{H} \cdot d\boldsymbol{s} = \int_{\Gamma} \boldsymbol{H} \cdot d\boldsymbol{s} = 4\pi i,$$

where the integral around $\Gamma$ is taken in the positive sense with respect to the region bounded by $\Gamma$, and the other integral is taken in the positive sense with respect to the region bounded by $G^*$. We think of $G$ as the limit of approximating curves $\{G^*\}$, where the maximum of the arc lengths goes to zero. We therefore say that (for the proper sense of **H**)

$$\int_{G} \boldsymbol{H} \cdot d\boldsymbol{s} = 4\pi i.$$

Thus the integral of $\boldsymbol{H} \cdot d\boldsymbol{s}$ around any curve $G$ enclosing the wire once is $4\pi$ times the current in the wire, it being assumed that $G$ is in a plane perpendicular to the wire. The same kind of argument shows that the integral of $\boldsymbol{H} \cdot d\boldsymbol{s}$ around a plane curve $G$ not enclosing the wire vanishes.

Suppose now that $G$ is not a curve in a plane perpendicular to the wire; in fact let $G$ be an arbitrary closed space curve. We can let $G$ bound a surface composed of triangles as used in Section 3.14 where the wire passes through one of these triangles in a plane perpendicular to the wire, if it passes through the curve $G$. In the following sum of integrals around triangles we exclude one triangle, and let this be the one through which the wire passes, if there is such a triangle. The sum of the integrals of $H \cdot ds$ around all the triangles in, for example, the positive sense with respect to each triangle is equal to this integral around the outer perimeter $G^*$ of the surface in the positive direction plus this integral in the negative direction around the triangle through which the wire passes, if there is such a triangle. It follows that the integral of $H \cdot ds$ around $G^*$ is $4\pi$ times the current $i$ through the wire if $G$ encircles the wire. If $G^*$ does not encircle the wire the integral of $H \cdot ds$ vanishes. Suppose that $G$ is the limit of curves of type $\{G^*\}$. It then follows that *the integral of* $H \cdot ds$ *around* $G$ *is zero if* $G$ *does not enclose the wire, and is* $4\pi i$ *for the current* $i$ *in the wire if it encloses the wire once.* The current must be taken in a direction that corresponds properly to the direction of integration around the curve $G$.

With each point of a conductor we can associate a *current density vector* $J$, where $J = \sigma E$ for the *conductivity* $\sigma$ of the conductor, and the electric intensity $E$. The direction of $J$ is the direction of current flow. If $J$ is constant for a region with area $A$, and is everywhere normal to this region, the current flowing across this region has the magnitude $|J|A$. Current density can be defined similarly for surface currents. Let $\theta$ be an angle between $J$ and a normal ray to a plane region $R$ with area $A$. Let $C$ be the boundary of $R$. If the region $R$ is small enough we would expect the current across $R$ to be traveling for practical purposes as if it were the current along a straight wire, or composed of currents along several parallel straight wires. We conclude that $4\pi|J|A\cos\theta$ is a good approximation to the integral of $H \cdot ds$ around $C$, where $|J|\cos\theta$ is evaluated at a properly chosen point of the region $R$. We therefore write

$$\int_C H \cdot ds = 4\pi \iint_R J \cdot d\Sigma.$$

Since any plane region $R$ can be broken up into smaller regions and such subdivisions can be performed so that the maximum chord of these regions goes to zero, we conclude that

$$\int_C H \cdot ds = 4\pi \iint_R J \cdot d\Sigma$$

for any such region, **H** being the magnetic intensity associated with **J**.

Assuming that Stokes's theorem applies, we have

$$\int_C H \cdot ds = \iint (\text{curl } H) \cdot d\Sigma = 4\pi \iint J \cdot d\Sigma.$$

Since this relation is valid for every plane region containing a point $P$, we conclude that at $P$

$$\operatorname{curl} \boldsymbol{H} = 4\pi \boldsymbol{J}. \tag{5.3}$$

With $\boldsymbol{J} = \sigma \boldsymbol{E}$ this gives

$$\operatorname{curl} \boldsymbol{H} = 4\pi\sigma \boldsymbol{E}.$$

Maxwell postulated the existence of a displacement current proportional to $\partial \boldsymbol{E}/\partial t$, and for which

$$\operatorname{curl} \boldsymbol{H} = k \frac{\partial \boldsymbol{E}}{\partial t}, \tag{5.4}$$

which is analogous to (5.3). Here $k$ is the dielectric constant of the medium. The validity of (5.4) was verified experimentally by Hertz.

The magnetic intensity $\boldsymbol{H}$ of (5.4) must not be confused with the magnetic intensity $\boldsymbol{H}$ of (5.3), nor with $\boldsymbol{H}$ of (5.5). Superimposing the magnetic intensities, the equations (5.3) and (5.4) can be replaced by

$$\operatorname{curl} \boldsymbol{H} = k \frac{\partial \boldsymbol{E}}{\partial t} + 4\pi \boldsymbol{J}. \tag{5.5}$$

Equation (5.5) is *another triple of Maxwell's equations* written in vector form as a single equation. We often write this equation with $\boldsymbol{J}$ replaced by $\sigma \boldsymbol{E}$.

The equations (5.2) and (5.5) with $\boldsymbol{J} = \sigma \boldsymbol{E}$ are equivalent to independent equations in the six functions representing the components of $\boldsymbol{H}$ and $\boldsymbol{E}$, and are enough with boundary conditions and initial conditions to determine $\boldsymbol{H}$ and $\boldsymbol{E}$. We therefore need to look no further for equations relating $\boldsymbol{H}$ and $\boldsymbol{E}$. *Maxwell postulated that equations* (5.2) and (5.5) *are the general equations relating the magnetic intensity* $\boldsymbol{H}$ *and the electric intensity* $\boldsymbol{E}$ *for homogeneous isotropic media.*

To complete the set of Maxwell's equations we must adjoin the relations

$$\begin{aligned} \operatorname{div} \boldsymbol{D} &= 4\pi\rho, \\ \operatorname{div} \boldsymbol{B} &= \operatorname{div} \mu \boldsymbol{H} = 0, \end{aligned} \tag{5.6}$$

where $\boldsymbol{D} = k\boldsymbol{E}$ (see equation 3.30). The last relation is due to the absence of true magnetic charges.

The Maxwell equations above are valid for homogeneous isotropic media which are at rest. At each point of an isotropic medium $k$ and $\mu$ are independent of direction. For homogeneous media $k$ and $\mu$ do not vary from point to point.

By vector analysis

$$\operatorname{curl} \operatorname{curl} \boldsymbol{E} = \operatorname{grad} \operatorname{div} \boldsymbol{E} - \Delta \boldsymbol{E}.$$

In the absence of volume charges we have $\rho = 0$ and $\operatorname{div} \boldsymbol{E} = 0$, so that

$$\operatorname{curl} \operatorname{curl} \boldsymbol{E} = -\Delta \boldsymbol{E},$$

We take the curl of both sides of equation (5.2) and differentiate both sides of (5.4) with respect to $t$. Making use of the relation

$$\frac{\partial \operatorname{curl} \boldsymbol{H}}{\partial t} = \operatorname{curl} \frac{\partial \boldsymbol{H}}{\partial t}$$

to eliminate $\boldsymbol{H}$ we obtain

$$\Delta \boldsymbol{E} = k\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{5.7}$$

Equation (5.7) is valid in homogeneous isotropic dielectrics and empty space. Eliminating $\boldsymbol{E}$ from equations (5.2) and (5.4) we have

$$\Delta \boldsymbol{H} = k\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2}, \tag{5.8}$$

which has the same form as (5.7). For the function $E_x$ representing the x-component of $\boldsymbol{E}$ the vector equation (5.7) yields

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - k\mu \frac{\partial^2 E_x}{\partial t^2} = 0. \tag{5.9}$$

There are identical equations for the other components of $\boldsymbol{E}$ and $\boldsymbol{H}$ with respect to an $xyz$-Cartesian coordinate system. The solution of (5.7) for spherical waves yields the Hertz result for radio transmission.

A justification for displacement currents may be seen from a consideration of a capacitor with capacitance $C_1$ in a circuit as in Figure 77. We have found, in the ideal case, that the electric field intensity $\boldsymbol{E}$ between the plates is proportional to the charge $\sigma_1$. Since the capacitor is inserted in a branch for which the current $i$ is $d\sigma_1/dt$, it is logical to associate a current with the term $\partial \boldsymbol{E}/dt$, proportional to this derivative.

## 5.2. Light.

Although it is necessary to go to quantum mechanics to explain certain phenomena, experiments indicate that light is electromagnetic and satisfies Maxwell's equations.

We shall concern ourselves with the important case of a plane wave, that is a wave for which all quantities depend on $t$ and one space variable only. Let this space variable be $x$, whence the partial derivatives of components of $\boldsymbol{E}$ and $\boldsymbol{H}$ with respect to $y$ and $z$ vanish. Suppose that $\rho = 0$ for the region of space considered. Since by (5.6), with $k$ constant,

$$\operatorname{div} \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0,$$

we have

$$\frac{\partial E_x}{\partial x} = 0.$$

Thus $E_x$ and $H_x$ are constants. Since they are constants, they are of no interest in the study of wave motion, and may be taken equal to zero.

By (5.7)

$$\frac{\partial^2 E_y}{\partial x^2} = k\mu \frac{\partial^2 E_y}{\partial t^2},$$
$$\frac{\partial^2 E_z}{\partial x^2} = k\mu \frac{\partial^2 E_z}{\partial t^2}$$

for the functions $E_y$ and $E_z$ representing components of $\boldsymbol{E}$. We can write these equations as

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 E_y}{\partial t^2}, \qquad \frac{\partial^2 E_z}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 E_z}{\partial t^2}, \tag{5.10}$$

where

$$v = \frac{1}{\sqrt{k\mu}}.$$

We now have a solution for $E_y$ of the form

$$E_y = E_0 \sin \omega \left(t - \frac{x}{v}\right), \tag{5.11}$$

where $\omega$ and $E_0$ are constants. This is readily verified by direct substitution.

There is a similar solution for $E_z$ with other arbitrary constants. We shall concern ourselves with $E_y$ only. Equation (5.4) now yields

$$\frac{\partial H_z}{\partial x} = -k \frac{\partial E_y}{\partial t}. \tag{5.12}$$

Substitution of $E_y$ given in (5.11) into (5.12) yields

$$\frac{\partial H_z}{\partial x} = -\omega k E_0 \cos \omega \left(t - \frac{x}{v}\right).$$

Integration gives

$$H_z = vkE_0 \sin \omega \left(t - \frac{x}{v}\right) + f(t)$$

for an arbitrary function $f(t)$. Substituting for $H_z$ in

$$\frac{\partial^2 H_z}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 H_z}{\partial t^2}$$

we obtain

$$\frac{\partial^2 f(t)}{\partial x^2} = \frac{1}{v^2} f''(t).$$

Thus $f''(t)$ vanishes, and

$$f(t) = At + B$$

for constants $A$ and $B$. By (5.2) we have

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t}.$$

Substitution for $E_y$ and $H_z$ yields $A = 0$. Since we are interested in waves we may drop the constant $B$. Thus

$$H_z = vkE_0 \sin \omega \left(t - \frac{x}{v}\right). \tag{5.13}$$

The expression $E_y$ is a function $E_y\ (x, t)$. As given in (5.11) we have

$$E_y(x, t) = E_y(x + v\Delta t, t + \Delta t)$$

for each set of values of $x$, $t$, and the increment $\Delta t$. This means that we have *a plane wave traveling with the speed $v$ in the positive direction of the $x$-axis.* Let $c$ be the speed of the wave in a vacuum. Then, no matter what units we use, we have

$$v = \frac{c}{\sqrt{k\mu}}.$$

This $v$ is then *the speed of light in the given medium.*

The vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are now given by $E_y$ and $H_z$, whence $\boldsymbol{E}$ and $\boldsymbol{H}$ are perpendicular, in phase, but with different formulas for amplitudes.

The plane wave given by (5.11) and (5.13) is of much practical interest. Thus at a considerable distance from the source a radio wave may be treated as plane. That this plane wave is rather general can be seen from the following considerations. We shall need some properties of series which will be stated but not proved. By definition we have a plane wave associated with $E_y$ if there is a number $\omega$ such that

$$E_y\left(x, t + \frac{2\pi}{\omega}\right) = E_y(x, t)$$

for all values of $x$ and $t$ under consideration. For each $x$ the function $E_y$ is then a periodic function of $t$ with the period $2\pi/\omega$. Consider a given value of $x$. If $E_y$ is a continuous function of $t$ for an interval of values of $t$ with length $2\pi/\omega$ or more, we can write $E_y$ as the unique *Fourier series*

$$E_y = a_0(x) + [a_1(x) \cos \omega t + b_1(x) \sin \omega t] + [a_2(x) \cos 2\omega t + b_2(x) \sin 2\omega t] + \ . \ . \ . \ ,$$

which converges to $E_y$. The $a$'s and $b$'s are functions of $x$ that can be expressed in terms of $E_y$ (note reference 5 at the end of this chapter). Assume that we can differentiate this series term by term to obtain unique series that converge to the first and second derivatives of $E_y$ with respect to $x$ and $t$. The first equation in (5.10) is then an equality between Fourier series. Transforming (5.10) to an equation with the non-vanishing terms all on one side we obtain a Fourier series equal to zero. Now zero has a unique Fourier series in sines and cosines of multiples of $\omega t$, and the coefficients of these sines and cosines vanish. Thus the terms in the Fourier series for $E_y$ in a given multiple of $\omega t$ must satisfy the equation (5.10). We can thus study the *harmonic components* of $E_y$ separately.

It is therefore no real restriction on the generality of the method to consider only a solution of (5.10) of the form

$$E_y = h(x) \sin \omega t + g(x) \cos \omega t.$$

By differentiation we obtain

$$\frac{\partial^2 E_y}{\partial x^2} = h''(x) \sin \omega t + g''(x) \cos \omega t,$$
$$\frac{\partial^2 E_y}{\partial t^2} = -\omega^2 E_y,$$

so that the first equation in (5.10) becomes

$$\left[h''(x) + \frac{\omega^2}{v^2} h(x)\right] \sin \omega t + \left[g''(x) + \frac{\omega_2}{v^2} g(x)\right] \cos \omega t = 0.$$

This implies that

$$h''(x) + \left(\frac{\omega}{v}\right)^2 h(x) = 0.$$

This ordinary differential equation has the general solution

$$h(x) = A \cos \frac{\omega x}{v} + B \sin \frac{\omega x}{v}$$

for constants $A$ and $B$. Similarly,

$$g(x) = C \cos \frac{\omega x}{v} + D \sin \frac{\omega x}{v}$$

for constants $C$ and $D$.

It follows that $E_y$ is a linear combination of the four products of the sine and cosine of $\omega x/v$ by the sine and cosine of $\omega t$, where the coefficients are arbitrary. By trigonometric identities we can then express $E_y$ as a linear combination of the sines and cosines of $\omega(t + x/v)$ and $\omega(t - x/v)$. The sine and cosine of $\omega(t + x/v)$ represent waves traveling toward the negative half of the $x$-axis, while the sine and cosine of $\omega(t - x/v)$ represent waves traveling toward the positive half of the $x$-axis. Since the term in the cosine of $\omega(t - x/v)$ represents a wave of the same type as the term in the sine of this angle, we may restrict ourselves to the sine only, which we have done in (5.11). Thus the solution (5.11) is not as special as it might first appear.

Visible light waves, heat waves due to radiation, and radio waves are examples of electromagnetic waves which satisfy Maxwell's equations. We apply the term *radiation* to such waves.

In the field of geometric optics it is assumed that light is composed of rays which travel in straight lines in a homogeneous isotropic medium, and that these rays are independent of each other. Let a ray $R$ strike a plane surface, and let $R_f$ be the reflected ray. The *angle of incidence* is the smallest angle between $R$ and the normal ray $N$ to the plane, where the normal is on the same side of the plane as $R$. The smallest angle between $N$ and $R_f$ is termed the *angle of reflection*. From Maxwell's equations and continuity assumptions we can justify the statement that *the angle of incidence is equal to the angle of reflection.* This is the *law of reflection.* If the ray $R$ strikes a surface at a point $P$ where the surface has a tangent plane, the ray $R$ strikes the surface as if it struck the plane tangent to this surface at $P$.

When a ray $R$ strikes a surface there may be a refracted ray $R_r$ traveling from the point of impact into the medium $M_2$ beyond the surface. By definition the *angle of refraction* is the smallest angle between $R_r$ and the ray normal to the surface at the point of impact and directed into the medium $M_2$ beyond the surface. Let the medium on the other side of the surface be called $M_1$. Maxwell's equations and continuity assumptions, can be used to justify the *law of refraction* of geometric optics, called Snell's law, which says that *the ratio of the sine of the angle of incidence to the sine of the angle of refraction is equal to the ratio $n_2/n_1$ of constants $n_1$ and $n_2$ of the media $M_1$ and $M_2$, respectively.* These constants are called *indices of refraction.* The index of refraction of a medium is the ratio of the speed of light in empty space to its speed in the medium.

**5.3. Energy density.** It is convenient to associate an energy density with the points of space at which we have an electromagnetic field. We shall arrive at this concept by means of dimensional analysis.

By the inverse square law we have

$$F = \frac{q^2}{\epsilon L^2}$$

for the dimensions $F$, $\epsilon$, $L$, and $q$ of force, permittivity, length, and electric charge. Thus $q$ has the dimensionality $F^{\frac{1}{2}}\epsilon^{\frac{1}{2}}L$. The electric field intensity $\boldsymbol{E}$ is a force per unit charge, and therefore has the dimensionality $F^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}L^{-1}$. The simplest scalar that can be obtained from $\boldsymbol{E}$ is probably $\boldsymbol{E} \cdot \boldsymbol{E}$. The dimensionality of $\epsilon\boldsymbol{E} \cdot \boldsymbol{E}$ is $\epsilon(F^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}L^{-1})^2$, which is $F/L^2$. Since energy has the dimensionality $FL$, energy density has the dimensionality $F/L^2$. We therefore say that an electric field has an *energy density* proportional to $\epsilon\boldsymbol{E} \cdot \boldsymbol{E}$.

Various justifications can be made for this definition of energy density. One of the simplest follows.

We consider a capacitor composed of two parallel plates (see Section 3.13). We suppose that these plates are a distance $d$ apart, that the area of a face is $A$, and that the charge density on this face is $\sigma$, whence the total charge $q$ of the capacitor is $A\sigma$. We let $C$ be the capacitance of the capacitor, so that

$$q = Cv \tag{5.14}$$

for the potential difference $v$ between the plates. By Section 3.13 the electric induction $\boldsymbol{D}$ satisfies

$$|\boldsymbol{D}| = 4\pi\sigma, \tag{5.15}$$

where we take $\sigma$ positive. Now

$$v = |\boldsymbol{E}|d, \tag{5.16}$$

where $\boldsymbol{E} = \boldsymbol{D}/k$ for the dielectric constant $k$. By (5.15) we have

$$q = \frac{A|\boldsymbol{D}|}{4\pi}. \tag{5.17}$$

By (5.14), (5.16), and (5.17) we have

$$C = \frac{A|\boldsymbol{D}|}{4\pi|\boldsymbol{E}|d} = \frac{Ak}{4\pi d}. \tag{5.18}$$

We imagine that a capacitor is charged by successively bringing charges from one plate to the other. Suppose that we transfer charge $\Delta q$ from one plate of the capacitor to the other. We then do work $\Delta W$, where

$$\Delta W = V^*\Delta q = \frac{q^*}{C}\Delta q$$

for the potential difference $V^*$ between the plates at some time during the addition of the charge $\Delta q$, and the value $q^*$ of the charge $q$ corresponding to the potential difference $V^*$. We add up such increments of work where we start with zero charge on the capacitor and build up the charge to a value $Q$. Taking the limit as usual in the calculus, we obtain the total work $W$ necessary to build up the charge $Q$, where

$$W = \frac{1}{C}\int_0^Q q\,dq.$$

If the potential difference for the charge $Q$ is $V$, we have

$$W = \tfrac{1}{2}CV^2, \tag{5.19}$$

which is then the potential energy stored in the capacitor. We imagine that this energy is uniformly stored in the space between the plates. Since $Q = A\sigma$ for the charge density $\sigma$ corresponding to $Q$, relations (5.18) and (5.19) give

$$W = \frac{2\pi\sigma^2Ad}{k}.$$

If we divide $W$ by the volume $Ad$ of the region between the plates and make use of (5.15) we have the energy density $u$, where

$$u = \frac{2\pi\sigma^2}{k} = \frac{|\boldsymbol{D}|^2}{8\pi k} = \frac{\boldsymbol{D}\cdot\boldsymbol{D}}{8\pi k}$$

It follows that

$$u = \frac{k|\boldsymbol{E}|^2}{8\pi} = \frac{k\boldsymbol{E}\cdot\boldsymbol{E}}{8\pi}$$

for the electric intensity $E$. This relation is taken as the definition of the *energy density* due to the electric field with the electric intensity $\boldsymbol{E}$.

We similarly define the energy density $u$ due to a magnetic field to be given by

$$u = \frac{\mu\boldsymbol{H}\cdot\boldsymbol{H}}{8\pi}$$

for the magnetic intensity $\boldsymbol{H}$ and the permeability $\mu$.

With both electric and magnetic fields present we thus have the energy density $u$, where

$$u = \frac{(k\boldsymbol{E}\cdot\boldsymbol{E} + \mu\boldsymbol{H}\cdot\boldsymbol{H})}{8\pi}. \tag{5.20}$$

**5.4. Radiation pressure.** To give a proof of the Stefan-Boltzmann law in Chapter 8 we shall need the concept of pressure of radiation, which we now derive.

Consider first a plane surface $\pi$ bounding two media $M_1$ and $M_2$. Let $STUV$ (Figure 101) be a rectangle with two sides parallel to the surface $\pi$, where the side $ST$ is in $M_1$ and the side $UV$ in $M_2$. The symbol $\psi$ designates the flux across this rectangle. Since the electromotive force $e$ for the circuit formed by the boundary of the rectangle is the integral of $\boldsymbol{E}\cdot d\boldsymbol{s}$ (in the direction shown) around the rectangle for the electric intensity $\boldsymbol{E}$, we have

$$e = \int_{ST} \boldsymbol{E}\cdot d\boldsymbol{s} + \int_{UV} \boldsymbol{E}\cdot d\boldsymbol{s} + \epsilon,$$

where $\epsilon$ is the contribution of the sides $TU$ and $SV$. For convenience take the bounding plane $\pi$ parallel to the $yz$-plane. By the first theorem of the mean of the integral calculus and the continuity of $\boldsymbol{E}$ in each medium

$$\int_{ST} \boldsymbol{E}\cdot d\boldsymbol{s} = E_{t1}L, \quad \int_{UV} \boldsymbol{E}\cdot d\boldsymbol{s} = -E_{t2}L,$$

where $L$ is the length of the side $ST$, and $E_{t1}$, $E_{t2}$ are functions representing the components of $\boldsymbol{E}$ tangential to the sides $ST$ and $UV$ respectively, evaluated at properly chosen points on these sides. By Maxwell's equations (see Theorem 3.8) the quantity $e$ is $-d\psi/dt$. Take the limit as the sides of the rectangle approach the surface simultaneously. Let $\boldsymbol{B}_n$ be the component of the flux density normal to the rectangle. We suppose that $|\boldsymbol{B}_n|$ is bounded by a number $M$. Let $A$ be the area of the rectangle. The flux across the rectangle is $\lambda A$, where $|\lambda| \leq M$. Now

$$\frac{d\psi}{dt} = A\frac{d\lambda}{dt}.$$

We assume that $d\lambda/dt$ exists and is bounded in absolute value. It follows that as the sides of the rectangle approach the surface $\pi$, the derivative $d\psi/dt$ goes to zero. Thus the limits of $E_{t1}$ and $E_{t2}$ are equal. Now taking the limit as $ST$ goes to zero, we find that *at each point of the bounding surface* $\pi$ *the tangential component of* $\boldsymbol{E}$ *is continuous.* It is seen that by limit methods we can show that this is true even though the bounding surface is not plane.

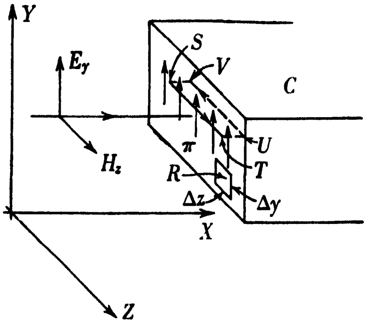

**Fig. 104. Plane wave striking a plane surface of a conductor $C$.**

We consider a plane wave in a vacuum striking the plane surface of an almost perfect conductor $C$ head-on as in Figure 104. We suppose that the surface of the conductor is in the plane $x = 0$. We assume that the wave has the intensities $\boldsymbol{E}$ and $\boldsymbol{H}$ given by

$$E_y = E_0 \sin \omega\left(t - \frac{x}{c}\right),$$

$$H_z = cE_0 \sin \omega\left(t - \frac{x}{c}\right),$$

as in Section 5.2, where $c$ is now the speed of the wave. The conductor is understood to have very large conductivity so that for practical purposes there is no current inside the conductor, and no electromagnetic field. The wave impinging on $C$ is then completely reflected. Since the electric intensity is zero inside the conductor, its tangential component is zero at the surface. We shall use primes to denote quantities associated with the reflected wave. The quantities $(E_y + E'_y)$ and $(E_z + E'_z)$ vanish at the surface, and we have

$$E'_y = -E_0 \sin \omega t, \quad E'_z = 0$$

at the surface. Since $E'_y$ satisfies Maxwell's equations, $E'_y$ must be of the form

$$E'_y = -E_0 \sin \omega \left(t + \frac{x}{c}\right).$$

The components of the field intensities are independent of $y$ and $z$. By (5.2) we have (noting that the units are taken so that the speed of the wave is $c$ in a vacuum and not 1)

$$H'_z = cE_0 \sin \omega \left(t + \frac{x}{c}\right).$$

Since $E'_z$ is continuous at the surface of $C$, and the limit of $E'_z$ is zero as the surface is approached, and further $E'_z$ satisfies Maxwell's equations, it follows that

$$E'_z = 0$$

everywhere. Outside of the conductor there is no current. By (5.1), (5.4), and (5.6)

$$H'_x = H'_y = E'_x = 0.$$

The reflected wave is now determined. We will observe that when a component of $\boldsymbol{E}'$ or $\boldsymbol{H}'$ turned out to be a constant, we wrote zero for it as we did in deriving (5.13).

From the manner in which magnetic fields arise we know [see equation (5.5)] that the incoming wave induces a current on the surface of $C$, and this current emits the reflected wave. Let $\boldsymbol{J}$ denote the density of this current. The magnetic field of $\boldsymbol{J}$ reinforces the incoming magnetic wave near the surface outside of $C$ and for practical purposes cancels it near the surface inside of $C$.

We shall compute the integral of $\boldsymbol{H}' \cdot d\boldsymbol{s}$ along the perimeter of the horizontal rectangle $STUV$ which was used at the beginning of this section. The rectangle $STUV$ has the side $ST$ near the surface of the conductor $C$ but outside the conductor and parallel to the $z$-axis, whereas the opposite side $UV$ is inside the conductor. Due to symmetry the sum of the integrals of $\boldsymbol{H}' \cdot d\boldsymbol{s}$ from $V$ to $S$ and $T$ to $U$ vanishes. Let the sides $ST$ and $UV$ be the same distance from the surface of $C$. Now $L$ is the length of the side $ST$. The integral of $\boldsymbol{H}' \cdot d\boldsymbol{s}$ around the rectangle is (approximately) $2|\boldsymbol{H}'|L$, that is, $2|\boldsymbol{H}|L$ for the magnetic intensity $\boldsymbol{H}$. Since the path around the rectangle encloses a current $|\boldsymbol{J}|L$, this integral is $4\pi|\boldsymbol{J}|L$. It follows that

$$|\boldsymbol{H}| = 2\pi|\boldsymbol{J}|$$

approximately.

We consider the element $R$ of area $\Delta y \Delta z$ as shown in the figure. The current on the surface of the conductor $C$ flows normal to the magnetic intensity $\boldsymbol{H}'$, and is thus in the $y$-direction. The current flowing across a line segment of length $\Delta z$ is then $|\,\boldsymbol{J}\,|\Delta z$. By Section 3.7 there is a force $\Delta\boldsymbol{F}$ on the surface element $R$, where

$$|\,\Delta\boldsymbol{F}\,| = |\,\boldsymbol{B}\,|\cdot|\,\boldsymbol{J}\,|\cdot\Delta A$$

for the area $\Delta A$ of the region $R$, and magnetic induction $\boldsymbol{B}$ corresponding to $\boldsymbol{H}$. This force is due to the pressure of the incoming wave and arises because of the reaction of the magnetic field of the incoming wave with the current induced by it on the surface of the conductor. For a vacuum (and the proper units) we have

$$|\,\Delta F\,| = |\,\boldsymbol{H}\,|\cdot|\,\boldsymbol{J}\,|\Delta A.$$

Let $\boldsymbol{H}''$ designate the total magnetic intensity $(\boldsymbol{H} + \boldsymbol{H}')$. The pressure $p$ on the element $R$ is by definition $|\,\Delta\boldsymbol{F}\,|/\Delta A$, whence

$$p = \frac{|\,\Delta\boldsymbol{F}\,|}{\Delta A} = \frac{|\,\boldsymbol{H}\,|^2}{2\pi} = \frac{|\,\boldsymbol{H}''\,|^2}{8\pi} = \frac{\boldsymbol{H}''\cdot\boldsymbol{H}''}{8\pi},$$

where $\boldsymbol{H}$ and $\boldsymbol{H}''$ are evaluated at the surface. Since the electric energy density is zero at the conductor surface, the pressure $p$ is the electromagnetic energy density $u$ at the surface (see Section 5.3). We take the limit as the conductor tends to a perfect conductor with infinite conductivity, and the side of the rectangle, along which we evaluated the integral of $\boldsymbol{H}'\cdot d\boldsymbol{s}$, approaches the surface of the conductor. We thus have

$$p = u \tag{5.21}$$

at the plane surface of a perfect conductor, where $u$ is the energy density at the conductor, and $p$ is the pressure of radiation associated with a plane wave normally incident with the surface of the conductor.

The radiation pressure is clearly the sum $(u_i + u_r)$ of the energy densities $u_i$ and $u_r$ of the incident and reflected waves respectively. In terms of the given waves, with the units chosen so that $\epsilon_0 = \mu_0 = c = 1$, the energy densities $u_i$ and $u_r$ are $(E_y^2 + H_z^2)/8\pi$ and $(E_y'^2 + H_z'^2)/8\pi$ respectively.

As a result of the above considerations we associate momentum with electromagnetic waves as well as energy density. We suppose that there is a *momentum density* at each point as well as energy density. Let $|\,\boldsymbol{g}_i\,|$ and $|\,\boldsymbol{g}_r\,|$ be the magnitudes of the densities $\boldsymbol{g}_i$ and $\boldsymbol{g}_r$ of the momenta of the incident and reflected waves respectively. Here $\boldsymbol{g}_i$ and $\boldsymbol{g}_r$ are respectively directed toward and away from the surface of the conductor $C$. In a time $\Delta t$ the magnitude of the momentum incident with the region $R$ of the surface of the conductor $C$ is $|\,\boldsymbol{g}_i\,|\Delta V$ for the volume $\Delta V = c\Delta A\Delta t$, and a properly chosen value $|\,\boldsymbol{g}_i\,|$. There is a similar expression for momentum carried away by the reflected wave. In the expression $|\,\boldsymbol{g}_i\,|\Delta V$ the absolute value $|\,\boldsymbol{g}_i\,|$ of the momentum density $\boldsymbol{g}_i$ is between the maximum and minimum values of the magnitude of the incident momentum density, assumed to be continuous, and striking $R$ in the time $\Delta t$. By Newton's second law of motion the force $\Delta F$ acting on the element $R$ has the magnitude $|\Delta F|$, where

$$|\Delta F| = \lim_{\Delta t \to 0} g^* \frac{\Delta V}{\Delta t},$$

and $g^*$ is $|g_r - g_i|$, that is, the sum of $|g_i|$ and $|g_r|$. Thus

$$|\Delta F| = cg^* \Delta A.$$

As a consequence

$$p = cg^*, \tag{5.22}$$

and

$$g^* = \frac{u}{c}. \tag{5.23}$$

Relation (5.21) is valid for surfaces that are not perfect reflectors, where $u$ is the total electromagnetic energy density at the surface.

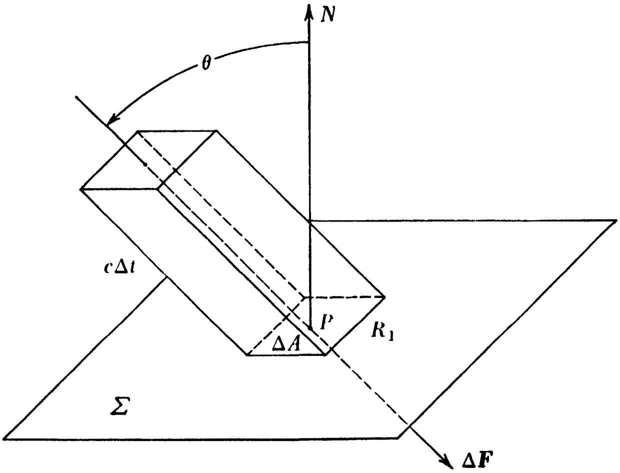

**Fig. 105.** Prismatic element making an angle θ with the normal $N$ to a surface Σ.

We shall consider radiation impinging on an element of a surface Σ, where the element has area $\Delta A$, and the radiation is at an angle $\theta$ with respect to the normal to the surface, the normal being directed toward the side of the surface from which the radiation is coming. For convenience we may consider this radiation as due to plane waves striking and leaving the surface at an angle $\theta$, where the surface is part of a plane. In a time $\Delta t$ the radiation in the prism shown in Figure 105 strikes the surface Σ, where the slant height of the prism is $c\Delta t$. At the same time radiation leaving the element in the direction $\theta$ due to reflection during the time $\Delta t$ fills this prism. The corresponding force acting on the surface element is $\Delta F$, where

$$|\Delta F| = cg^* \cdot (\cos\theta)\Delta A,$$

which can be written as

$$|\Delta F| = u(\cos\theta)\Delta A,$$

for properly chosen values of $g^*$ and $u$. The direction of the force $\Delta F$ is into the body of which $\Sigma$ is the surface, and makes an angle $(180° - \theta)$ with the normal ray $N$. This force has normal and tangential components represented by $\Delta F_n$ and $\Delta F_t$ respectively, where

$$\begin{aligned} \Delta F_n &= u(\cos^2\theta)\Delta A, \\ \Delta F_t &= u(\cos\theta\sin\theta)\Delta A. \end{aligned} \tag{5.21}$$

Taking the limit, as $\Delta t$ and the maximum chord of the element go to zero simultaneously, while the element shrinks to a point $P$ on $\Sigma$, we obtain the normal pressure $p$ given by

$$p = u\cos^2\theta, \tag{5.25}$$

and a tangential stress $p_t$ given by

$$p_t = u\cos\theta\sin\theta. \tag{5.26}$$

The density of the electromagnetic field at a point $P$ varies at $P$ with the wave. It will be understood that we should use an average value of $u$, and corresponding average values of $g^*$ and $p$.

It is convenient to imagine that radiant energy is localized in separate beams or rays. Through one point in space there may be infinitely many rays, but there can be only one ray through two given points. We suppose that the rays are traveling in a homogeneous and isotropic medium, so that they are straight, and the velocity of propagation is the same in all directions. Let $L$ be a straight line that passes through the plane regions $R_1$ and $R_2$ in space. We let $A_1$ and $A_2$ be the areas of these regions, and let $\theta_1$ and $\theta_2$ be smallest positive angles that the normals to these regions make with the line $L$. We suppose that the distance $r$ from a point on $R_1$ to a point on $R_2$ is large compared to the maximum chords on $R_1$ and $R_2$. By assumption (which we shall justify presently), the rate at which radiant energy is transmitted from $R_1$ to $R_2$ is for practical purposes the quantity $Q$, where

$$Q = K\frac{A_1A_2\cos\theta_1\cos\theta_2}{r^2} \tag{5.27}$$

and $K$ is a constant. It is assumed that $Q$ in (5.27) is also the rate at which radiant energy is transmitted from the region $R_2$ to the region $R_1$, the radiation being associated with the same fourfold infinity of straight lines joining points of $R_1$ to points of $R_2$. Relation (5.27) is approximately valid if $R_1$ and $R_2$ are almost plane. In the limits that we shall take later the approximations disappear.

It is readily seen that the assumption of the validity of (5.27) is a reasonable one. Let a conical beam with vertex $V$ cut out regions $R_1$ and $R_2$ on spheres $S_1$ and $S_2$ with centers at $V$. Let $r_1$ and $r_2$ be the radii of the spheres $S_1$ and $S_2$ respectively, and let $A_1$ and $A_2$ be the respective areas of the regions $R_1$ and $R_2$. The same energy falls on the region $R_1$ as on the region $R_2$. If the radius $r_2$ is much larger than $r_1$, the radiation on $S_2$ is much weaker than on $S_1$; in fact, from solid geometry we know that the area cut out by the beam on a sphere with center at $V$ increases as the square of the radius. On the sphere $S_1$ a region with area $kA_1$ will have $k$ times the radiation falling on $R_1$. We therefore conclude that the radiation on a region $R_1$ is directly proportional to the area $A_1$ of $R_1$, and inversely proportional to the square of the radius $r_1$.

If the cone is narrow enough we may approximate the regions $R_1$ and $R_2$ on $S_1$ and $S_2$ by plane regions normal to a ray of the beam and with approximately the same areas as $S_1$ and $S_2$ respectively. Consider such a plane region $\pi$, and let $\pi'$ be a region in a plane that makes an angle $\theta$ with the plane of the region $\pi$. Suppose that the region $\pi'$ is such that it projects into the region $\pi$ using the rays of the beam to perform the projection. Suppose that all points of the region $\pi'$ are about the same distance from $V$ as the points of the region $\pi$. Approximately the same energy will then fall in a time $\Delta t$ on $\pi'$ as on $\pi$. The area of the region $\pi$ is now the area of the region $\pi'$ multiplied by $\cos\theta$. It follows that the radiant energy rate for $\pi'$ is approximately proportional to $A(\cos\theta)/R^2$ for the area $A$ of $\pi'$ and the distance $R$ from a point on $\pi'$ to $V$. The presence of the factor $A_2(\cos\theta_2)/r^2$ in (5.27) is now explained.

We return to the regions $R_1$ and $R_2$ in the discussion just above equation (5.27). Since $Q$ depends on $R_1$, we can now write

$$Q = f(R_1)\,\frac{A_2\cos\theta_2}{r^2}$$

for a function $f(R_1)$ of $R_1$. The radiation rate from $R_2$ to $R_1$ is the same as the radiation rate $Q$ from $R_1$ to $R_2$. Therefore $A_1(\cos\theta_1)/r^2$ is a factor of $Q$. We conclude that

$$f(R_1) = KA_1\cos\theta_1,$$

so that we arrive at (5.27).

The constant $K$ in (5.27) is often termed the *brightness* of the beam from $R_1$ to $R_2$. In what follows we shall restrict ourselves to *isotropic radiation* where the radiation is independent of time, is the same for all directions, and is the same at each point. The brightness $K$ is then independent of the point, direction, and time.

We consider a region $R_1$ with area $\Delta A$ as in Figure 106 where the points of $R_1$ are near the center of a sphere $\Sigma$ of radius $r$. We choose an element $R_2$ of area $\Delta\Sigma$ on the sphere $\Sigma$, where the rays leaving $R_1$ for $R_2$ make or approximately make an angle $\theta$ with a normal ray $N$ to $R_1$ as shown. The energy transported across the region $R_1$ in the direction $\theta$ and in a time $\Delta t$ fills, or approximately fills, a prism of slant height $c\Delta t$, where $c$ is the speed of propagation. The prism is pictured in Figure 105. The volume of the prism is $c \cos \theta \Delta t \Delta A$. By (5.27), the energy is equal, or approximately equal, to $Q\Delta t$, where

$$Q = \frac{K \cos \theta \Delta A \Delta \Sigma}{r^2}.$$

The corresponding contribution $\Delta u$ to the energy density $u$ at a point of the region $R_1$ is then approximately $Q\Delta t$ divided by the volume of the prism, whence

$$\Delta u = \frac{K\Delta\Sigma}{cr^2}. \tag{5.28}$$

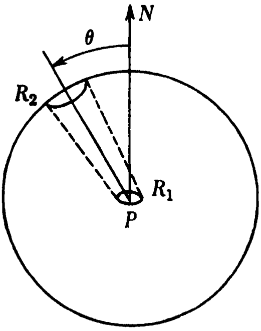

**Fig. 106. Source of radiant energy at the center of a sphere.**

The limit of (5.28) as $\Delta t$ tends to zero is an expression of the same type. We take the limit of the sum

$$\frac{K}{r^2c}\sum_{i=1}^{n} \Delta\Sigma_i$$

over the regions of a subdivision of the surface of the sphere $\Sigma$, as the maximum chord of the regions on the sphere tends to zero (in length) and simultaneously the maximum chord of $R_1$ tends to zero while $R_1$ shrinks to the center $P$ of the sphere $\Sigma$. We obtain the energy density

$$u = \frac{K}{r^2c}\iint_{\Sigma} \Delta\Sigma = \frac{4\pi K}{c}$$

at the point $P$. Thus

$$K = \frac{cu}{4\pi}. \tag{5.29}$$

Suppose now that the region $R_1$ of area $\Delta A$ in Figure 106 is a plane region on a perfectly reflecting surface of a given body. The beam traveling from the region $R_2$ on the sphere contributes an energy density $\Delta u_1$ to the density at the point $P$ of the region $R_1$, where by (5.28)

$$\Delta u_1 = \frac{K\Delta\Sigma}{cr^2}.$$

The radiation reflected from the surface at $R_1$ to the region $R_2$ on the sphere contributes an energy density $\Delta u_2$ to the density at the point $P$, where

$$\Delta u_1 = \Delta u_2.$$

The total contribution to the energy density at the point $P$ due to radiation between $R_1$ and $R_2$ is thus $\Delta u$, where

$$\Delta u = \Delta u_1 + \Delta u_2 = \frac{2K\Delta\Sigma}{cr^2}.$$

By (5.25) there is a normal pressure $\Delta p$ corresponding to the energy density $\Delta u$, where

$$\Delta p = \Delta u \cos^2 \theta.$$

Let $\Sigma^*$ be the hemisphere of radius $r$ with center at the point $P$ on the region $R_1$, and lying above $R_1$ with $R_1$ in its base. The normal pressure $p$ on $R_1$ is now the integral

$$p = \frac{2K}{cr^2} \iint\limits_{\Sigma^*} \cos^2 \theta d\Sigma.$$

Due to symmetry the stress $p_t$ tangent to $R_1$ vanishes.

We set up an $xyz$-Cartesian coordinate system with origin at the center $P$ of the hemisphere $\Sigma^*$, and the normal $N$ to $R_1$ at $P$ on the $z$-axis. For the hemisphere we have the equation

$$z = \sqrt{r^2 - x^2 - y^2},$$

from which

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \frac{r}{z}.$$

Further,

$$\cos \theta = \frac{z}{r}$$

for a point $(x, y, z)$ on the hemisphere. Thus

$$\iint\limits_{\Sigma^*} \cos^2 \theta d\Sigma = \frac{1}{r} \int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} z dx dy.$$

To evaluate the integral it is convenient to change to polar coordinates $\rho$ and $\phi$ from $x$ and $y$ so that

$$\iint_{\Sigma^*} \cos^2 \theta d\Sigma = \frac{1}{r}\int_0^{2\pi}\int_0^r \sqrt{r^2-\rho^2}\,\rho d\rho d\phi.$$

It follows that

$$\iint_{\Sigma^*} \cos^2 \theta d\Sigma = \frac{2\pi r^2}{3}. \tag{5.30}$$

Making use of $K$ in (5.29) and the relation (5.30) we find that

$$p = \frac{u}{3}.$$

Thus we have the following theorem.

***Theorem* 5.1.** *For isotropic radiation in a vacuum the pressure of radiation is one third of the energy density.*

The proof of this result depended on the fact that the radiation was due to plane waves. However, at some distance from a point source, or source region, a wave from this source is plane for practical purposes.

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4. Max Planck, *Introduction to Theoretical Physics*, Macmillan, London, 1932, Vol. III.
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# Problems for Part II

The reader is expected to transform the following electric-magnetic-mechanical systems into mathematical form; he is not expected to solve the equations so obtained.

1. A circuit is made up of resistance, capacitance, inductance, and a battery in series. The battery is suddenly short-circuited. We are interested in the current in the circuit.
2. Current leaks through the dielectric of a capacitor.
3. A generator supplies 110 alternating volts to an iron-cored transformer, in turn supplying a resistance load as in Figure 107. Take into account distributed capacitance between the primary and secondary of the transformer. We are interested in the load current.

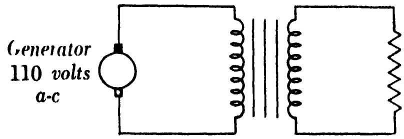


**Fig. 107. A simple generator circuit.**

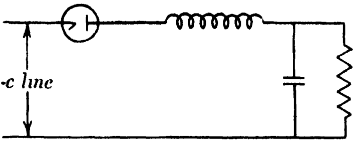


**Fig. 108. Half wave rectifier stage.**

4. Alternating current is fed to a circuit as shown in Figure 108 composed of a diode followed by a filter. We are interested in the output of the filter.
5. An arbitrary potential difference $e_i$ is applied to the bridge of Figure 109, and the potential difference $e_0$ is desired.
6. A varying voltage $e$ is applied to a solenoid with an iron plunger as in Figure 110. The plunger is spring loaded. Its position is desired.
7. A 220-volt alternating current generator supplies a resistance load through a long transmission line as in Figure 111
8. An electron moves in a fixed magnetic field.
9. An electron moves between charged plates.
10. Find the field of an electrically charged sphere.
11. Voltage sources $e_1$ and $e_2$ are applied to the grid circuit of a triode as in Figure 112. The output voltage $e_0$ is desired for alternating current inputs, and for arbitrary input voltages.
12. The push-pull circuit of Figure 113 is symmetrical with respect to the triodes. The output voltage $e_0$ for a given alternating or direct input voltage $e_i$ is desired.

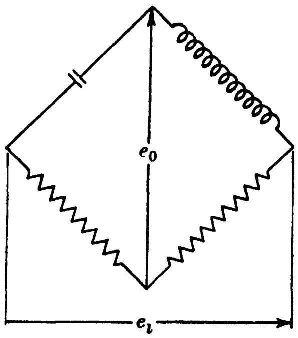


Fig. 109. Bridge.

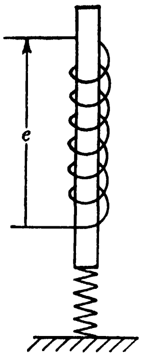


Fig. 110. Solenoid with a plunger.

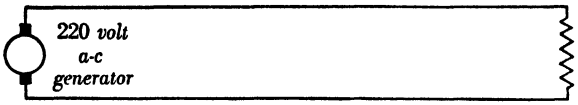


Fig. 111. Transmission line with an end load.

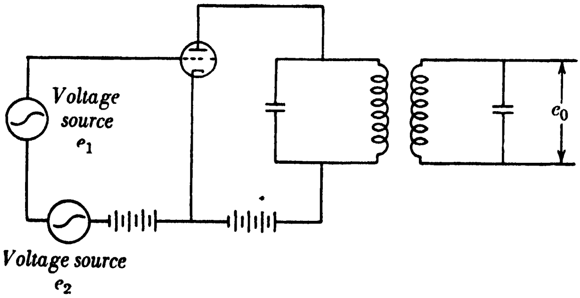


Fig. 112. Grid bias modulation.

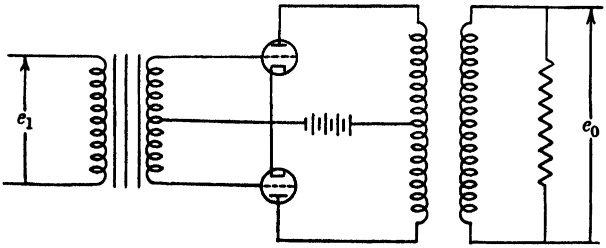


Fig. 113. Push-pull network with triodes.

13. An automobile driver suddenly applies his foot to a self-starter pedal. The speed of the crankshaft is desired.
14. An iron plate pivoted at one end, and spring loaded, is attracted by the poles of a fixed magnet as in Figure 114. The position of the plate is desired if it is initially set into motion by a blow.

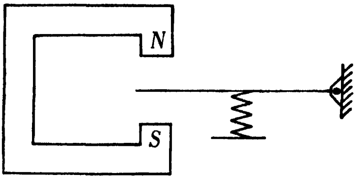


**Fig. 114. Metal plate attracted by a magnet.**

15. The voltage supplying a series motor driving the wheels of a locomotive is increased linearly with respect to time. The speed of the wheels and the instantaneous current in the motor are desired.
16. A horizontal compass needle is suspended from a steel wire with one end of the wire fixed as in Figure 115. The needle is in a fixed field. The needle is twisted through a given angle and suddenly released.

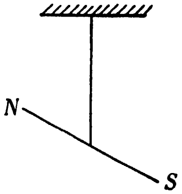


**Fig. 115. Compass needle suspended by a wire.**

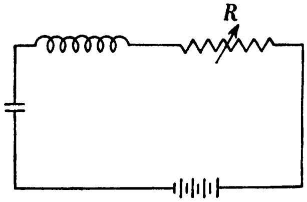


**Fig. 116. Circuit with a varying resistance.**

17. The resistance in the circuit of Figure 116 is varied linearly with respect to time.
18. In the circuit of Figure 117 the switch is first closed in the position 1 and then suddenly opened.

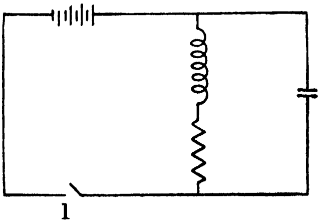


**Fig. 117. Circuit with a switch.**

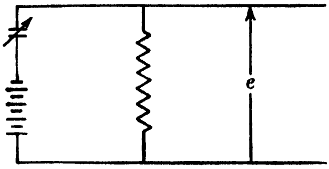


**Fig. 118. Variable capacitor in a circuit.**

19. A switch is placed in the transmission line of Figure 111 and is suddenly opened.

**20.** The capacitance of a capacitor in a microphone is varied sinusoidally in the circuit of Figure 118. The problem is to obtain the output voltage $e$.

**21.** In the circuit of Figure 116 with the capacitor omitted the value of the resistor is changed instantaneously from one value to another.

**22.** In the circuit of Figure 116 without the capacitor all elements are fixed during a transient except the inductance which is changed instantaneously from one value to another.

**23.** The potential across a capacitor builds up in the circuit of Figure 119 until it is high enough to discharge across an arc joining the terminals shown. Assume a negative slope for the volt-ampere curve of the arc.

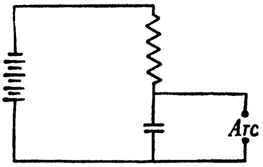


**Fig. 119. Capacitor discharging through an arc.**

**24.** The output $e_0$ of the band-pass filter of Figure 120 for a given input $e_i$ is desired.

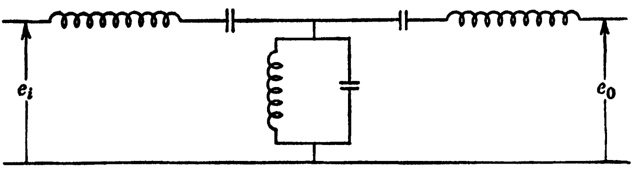


**Fig. 120. Band-pass filter.**

**25.** In the *three-phase* system of Figure 121 the branches 1, 2, and 3 are identical as to the values of the constants, the quantity $e$ being a given generator voltage. The branches 4, 5, and 6 are also identical.

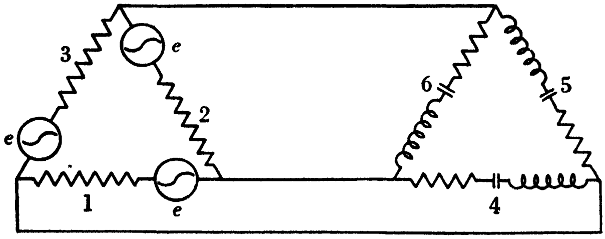


**Fig. 121. Three phase system.**

**26.** An alternating current generator is connected to a power line. A diesel engine driving the generator, shown in Figure 122, is suddenly shut down. For simplicity assume that the generator field is fixed.

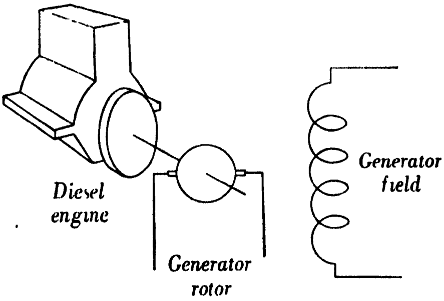


**Fig. 122. Generator driven by a diesel engine.**

27. The resistance of a photoelectric cell is inversely proportional to the light flux on the cell. For the circuit of Figure 123 the solenoid plunger movement is desired when the light intensity is suddenly changed. The resistance amplifier 1234 merely delivers an output voltage across the terminals 3 and 4 directly proportional to the voltage across the terminals 1 and 2. The solenoid plunger is spring-and-dashpot loaded as shown. The plunger is iron.

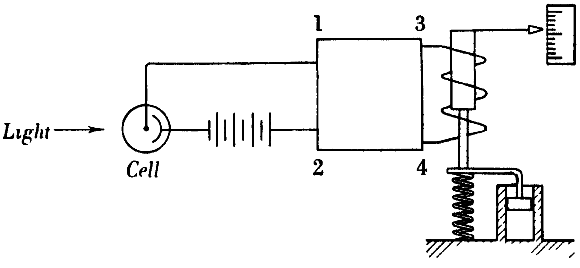


**Fig. 123. Circuit with a photoelectric cell.**

28. An infinitely long straight wire carrying a current $i$ is a distance $d$ from a magnetic pole $m$. By dimensional analysis obtain a formula for the magnitude $F$ of the force on $m$, assuming that the pole is a point pole.
29. A sinusoidal voltage is applied at one end of a transmission line. By dimensional analysis find a formula for the velocity of propagation of the disturbance.
30. Suppose that a non-linear inductance has an iron core for which the $BH$-curve can be represented by $i = k\psi^7$, where $i$ is the current through the inductance and $\psi$ is magnetic flux. A resistance $r$ is in series with the coil, and a constant potential difference $e_0$ is applied to the system at $t = 0$, where $t$ is time. By dimensional analysis find a formula for the instantaneous current in the coil.

# PART III

# HEAT

## CHAPTER

## 6

# *First law of thermodynamics*

**6.1. Temperature.** In Part III we shall be concerned with setting up equations for heat problems.

Through our sense of touch we experience warmth and cold in external objects. When a body becomes warmer we say that some *heat* has been added to it. Our observations show that a uniform body under constant pressure changes volume when heated. By *uniform* we mean that the body has the same chemical and physical properties at each point, where we use *point* in the large sense. The body is an aggregate of atomic particles that give us an average effect if the region of space considered is not too small. The term *small* here depends on the particular situation treated, especially on the size of the body under consideration. To be rigorous we may define a body to be uniform if the finest measurements we are using in our particular study show the body to be the same throughout.

Many substances, such as air, water, and solid steel, expand on being heated. This affords a simple method of determining whether or not one body is warmer than another.

It is our experience that when a warm body $A$, such as hot lead, is placed in contact with a colder body $B$, such as ice, the body $B$ becomes warmer and the body $A$ colder until these bodies reach a state in which we can recognize no change. If each of these bodies is uniform when a state of no change is reached, the bodies are said to be in *thermal equilibrium*, and to have the same *temperature*. It is evident that absolute thermal equilibrium is not possible, but can be approximated to great accuracy in the laboratory.

*Thermodynamics* is the study of thermal equilibria of physical systems. The following law is based on a tremendous amount of physical evidence.

***Law* 6.1. (Zeroth law of thermodynamics).** *If a body A is in thermal equilibrium with bodies B and C, the bodies B and C are in thermal equilibrium with each other.*

This law enables us to tell whether or not bodies $B$ and $C$ are at the same temperature by comparing them with a standard $A$. We can tell whether or not $B$ is hotter than $C$ by the volume of $A$ when $B$ and $C$ are separately placed in contact with $A$. The body $A$ might be, for example, the mercury in a glass thermometer.

To treat temperature quantitatively, it is necessary to have a unit of measure. In the case of the centigrade scale the difference in the volume $V_0$ of $A$ in thermal equilibrium with melting ice under atmospheric pressure and the volume $V_{100}$ of $A$ in thermal equilibrium with boiling water at atmospheric pressure is said to correspond to 100 degrees of temperature difference. For convenience the temperature for the first state is said to be zero degrees *centigrade*, abbreviated 0 C, and the second state 100 degrees centigrade. For Fahrenheit the corresponding temperatures are 32 degrees *Fahrenheit* abbreviated 32 F, and 212 degrees Fahrenheit. Generally we shall let $\theta$ designate temperature in this chapter unless the temperature is *absolute*, to be defined later. The standard body used for measuring temperature is a *thermometer*. If we let $V$ be the volume of the body at $\theta$ degrees centigrade and constant pressure we have

$$\theta = f(V)$$

for a single-valued continuous function $f(V)$ of $V$. The nature of the function $f(V)$ is otherwise at our disposal. We therefore choose the simplest function $f(V)$ possible, namely a linear one, whence

$$\theta = f(V) = \frac{100(V - V_0)}{V_{100} - V_0}. \tag{6.1}$$

Formula (6.1) depends in general on the body used. If, however, we restrict ourselves to gases, (6.1) yields the same temperature $\theta$ over a large range of values to considerable accuracy. Thus for 50 to 200 degrees centigrade the gases hydrogen, helium, neon, nitrogen, air, argon, and oxygen can be used to give the same temperature to 0.1 per cent.

Since (6.1) depends on the body used, a more rigorous definition of temperature is required. It has been found experimentally that $\theta$ given by

$$\theta = \lim_{p\to 0} \frac{100(V - V_0)}{V_{100} - V_0} \tag{6.2}$$

is independent of the gas used, $p$ being the pressure of the gas. In Chapter 7, we shall show how a definition of temperature can be made on the basis of thermodynamics, independent of the substance used.

Since by experiment

$$\lim_{p\to 0} \frac{V_0}{V_{100} - V_0}$$

is the same for all gases, the quantity $T$ given by

$$T = \lim_{p\to 0} \frac{100V}{V_{100} - V_0} \tag{6.3}$$

is the same for all gases. The quantity $T$ is called the *absolute gas temperature.* The melting point of ice under standard conditions is then approximately 273 degrees absolute (abs); or what is the same thing, 0 degrees centigrade is about 273 degrees absolute.

Since Fahrenheit temperature $\theta_f$ is $32 + (\frac{9}{5})\theta$ degrees for $\theta$ in centigrade, *absolute Fahrenheit temperature* $T$ is approximately 460 plus Fahrenheit temperature $\theta_f$. Absolute temperature Fahrenheit is commonly called *Rankine.*

Temperature can be measured in other ways than volumetrically. For example, we can use the pressure of a gas while the volume is kept constant, the electric resistance of a platinum wire, or the potential difference established at the junction of two dissimilar metals of a *thermocouple.*

Temperature is related to the motion of the molecules in the given substance.

**6.2. Equation of state.** Experimental evidence shows that a liquid or gas in thermal equilibrium satisfies an *equation of state*

$$f(p, V, T) = 0 \tag{6.4}$$

for a function $f(p, V, T)$ of the pressure $p$, volume $V$ occupied by the given mass of fluid, and the absolute temperature $T$. For fluids the molecules are relatively free to move about, so that it is reasonable to expect a relation (6.4) to be valid. In problems involving those solids, where for practical purposes the only stress acting is a normal stress $p$ uniform in all directions, we may use an equation of state (6.4).

Gibbs' function will be explained in Section 7.7. A *phase* is a homogeneous system (chemically and physically identical at each point) composed of any number of chemical constituents, such as hydrogen and oxygen, having the same Gibbs' function everywhere. The well-known phases of a substance are the liquid, solid, and gaseous forms, but there may, for example, be several solid and liquid phases, as for solutions and alloys.

A *mole* of a substance is that quantity of a substance whose weight is the molecular weight of the substance. Thus if we use pounds, a mole of oxygen weighs 32 pounds. If one mole of a gas is at the pressure $p$ and occupies the volume $v$ we have the following form of the equation of state (6.4):

$$pv = RT, \tag{6.5}$$

where $T$ is the absolute temperature of the gas, and $R$ is the universal *gas constant.* For pressures up to 200 pounds per square inch (psi) the law (6.5) can be used to considerable accuracy for engineering purposes provided that the absolute temperature is at least twice the critical temperature, where the *critical temperature* is the temperature above which a gas can be compressed indefinitely without causing the gas to change to another phase. If the temperature of the gas is below the critical temperature an increase in pressure will eventually cause the gas to condense. For air at 32 F the error in $v$ computed from (6.5) is about 0.1 per cent at one standard atmosphere, and about 1 per cent at 300 atmospheres.

Equation (6.5) is the *general gas law* valid for *perfect gases*, also called *ideal* gases. The constant $R$ is 1545 foot pounds per pound-mole per degree Rankine.

That the constant $R$ must be the same for all gases can be proved by assuming the validity of (6.5) where $R$ may depend on the gas, and *Avogadro's law*, which states that all *perfect gases at a given pressure and temperature have the same number of molecules in any given volume*. Thus suppose that we have one mole of a gas at the pressure $p$ and temperature $T$, whence by (6.5)

$$pv_1 = R_1T$$

for the volume $v_1$ occupied by the gas, and constant $R_1$ for the gas. For a mole of a second gas we have

$$pv_2 = R_2T.$$

Since $v_1 = v_2$, we have $R_1 = R_2$, so that the independence of $R$ in (6.5) of the gas is proved.

Suppose now that we have $n_1, n_2, \ldots, n_r$ moles of perfect gases $A_1$, $A_2, \ldots, A_r$ thoroughly mixed, and occupying a volume $V$ at a pressure $p$ and temperature $T$. By (6.5)

$$pV = (n_1 + n_2 + \cdots + n_r)RT. \tag{6.6}$$

This is then the equation of state of the mixture. We are assuming here that the gases are not reacting with each other while mixed.

We can write (6.6) in the form

$$p = \frac{n_1}{V}RT + \frac{n_2}{V}RT + \cdots + \frac{n_r}{V}RT,$$

where the first term on the right is the pressure which the gas $A_1$ would exert if it filled the volume $V$ alone, the second term is this pressure for the gas $A_2$, etc., whence *the pressure p is the sum of the partial pressures exerted by the gases*. This relation between pressures is called *Dalton's law*.

A more accurate equation of state for a gas is the following due to van der Waals:

$$p = \frac{R'T}{v - b} - \frac{a}{v^2},$$

where $a$ and $b$ are small constants and $R'$ depends somewhat on the gas.

Consider now the equation

$$pv = R^*T,$$

where $p$ is in pounds per square foot, the quantity $v$ is the volume in cubic feet (specific volume) occupied by one pound of the gas, the temperature

$T$ is in degrees Rankine, and $R^*$ is a constant. Here $R^*$ is 53.3 for air, 48.3 for oxygen, 767 for hydrogen, 85.8 for steam at low pressures and temperatures, 35.1 for carbon dioxide, and 96.4 for methane. These typical values of $R^*$ can be computed from the universal gas constant $R$ of (6.5), and the molecular weights involved; $R^*$ is $R$ divided by the molecular weight of the gas.

With the aid of statistical thermodynamics we can derive an equation $p = f(v, T)$ for many substances.

From physical experience we know that two of the variables $p$, $V$ and $T$ of equation (6.4) can generally be taken as independent in the sense that the third variable is explicitly determined by (6.4). We shall assume that the regions in which we work are such that the equation of state can be solved uniquely for any one of the variables $p$, $V$ (or $v$), and $T$ in terms of the others. We also suppose that each curve on the $pV$-plane in the regions concerned corresponds to an actual physical process.

In many theoretical considerations it is convenient to take $p$ and $T$ as the independent variables, particularly because when two phases in contact are in thermal equilibrium, $p$ and $T$ are everywhere the same.

If conditions are changing rapidly, the variables $p$, $V$, and $T$ may vary over the body treated so that an equation of state (6.4) does not hold in the strict sense. We can often assume that at each instant the body is in thermal equilibrium, or that the body can be treated as an aggregate of subbodies, each in thermal equilibrium. If sufficiently accurate results are not obtained by this procedure, the development should be modified where necessary. In order to simplify the mathematics the analyst should assume whenever possible that his gas is always in thermal equilibrium, and that (6.5) holds. This is even true in the case of the gases in internal combustion engines, such as the gasoline engine, diesel, and gas turbine, and of steam in steam engines.

### 6.3. Pressure-volume diagram of water.

A curve on the $pV$-plane of a substance, for which the temperature is constant, is termed an *isotherm*. In Figure 124 isotherms of water are given. A gas near condensation to a liquid or solid is called a *vapor*. The reader will observe vaporization lines representing equilibrium between liquid and vapor phases. These lines become shorter as the temperature rises to the critical temperature above which there is no distinction between the liquid and vapor phases. The isotherm at the critical temperature is called the *critical isotherm*. Well above the critical isotherm the substance behaves like a perfect gas, since the isotherms approach equilateral hyperbolas. As we move along an isotherm to the left of the vapor saturation curve condensation takes place. To the left of the liquid saturation curve there is only liquid water. Above the vapor saturation curve we have superheated steam.

**6.4. Heat.** From experience we know that when two substances in contact are not in thermal equilibrium with each other, energy is transferred from one to the other until thermal equilibrium is reached. This energy is called *heat.* Heat travels from the body at the higher temperature to the body at the lower temperature. To measure the heat given out or taken on by a body we take the change in temperature experienced by a standard substance like water when this substance is the only substance in contact with the given body, and the changes are due to contact only. The *mean calorie* is defined to be 1/100th of the heat required to raise 1 gram of water from 0 degrees centigrade to 100 degrees centigrade at constant atmospheric pressure.

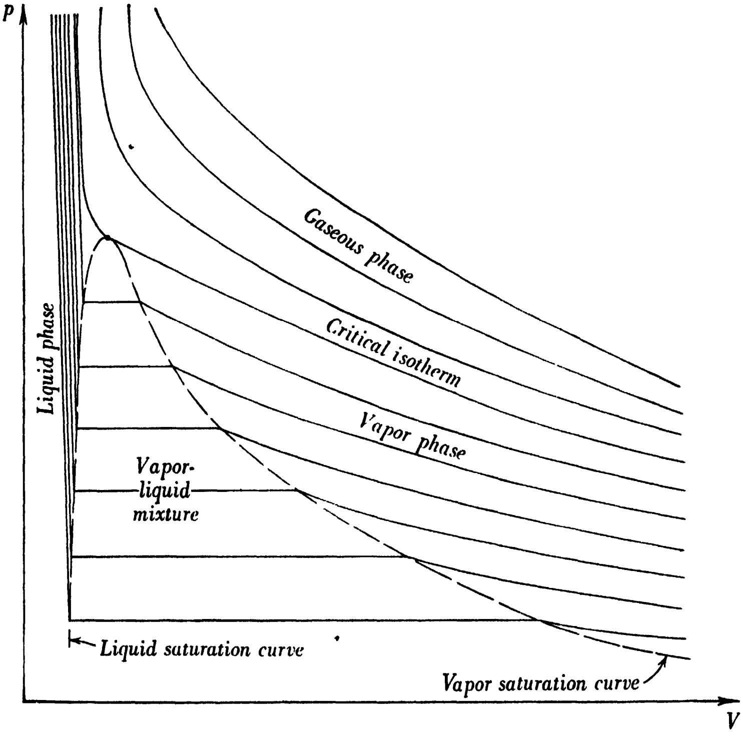


**Fig. 124. Pressure-volume diagram of water.**

We shall give an illustration of the heat given off by a body. A body is thermally *insulated* from its environment if the thermal equilibrium of the body is unaffected by states of substances beyond the insulating material. Of course, true insulation does not exist, but can be approximated very well in practice. Experiment shows that 1 gram of water at 0 degrees centigrade is heated at 1 atmosphere to 100 degrees centigrade by 10 grams of steel immersed in the water, where the steel is initially at 200 degrees centigrade, the system being thermally insulated from its environment. The 10 grams of steel thus lose 100 mean calories in cooling from 200 degrees centigrade to 100 degrees centigrade.

Joule proved the equivalence of heat to mechanical energy by experiments involving friction. In engineering the *British thermal unit* is preferred to the calorie, the British unit being abbreviated Btu. One British thermal unit is the quantity of heat required to raise the temperature of one pound of water from 63 degrees Fahrenheit to 64 degrees Fahrenheit at constant atmospheric pressure, and is approximately 778.26 foot-pounds, or 252 calories.

Suppose now that the temperature of a given mass of a substance is changed by an amount $\Delta T$ through some process, and that this corresponds to an amount of heat $\Delta Q$ taken on or given off by the substance depending on the sign of $\Delta T$. We form the quotient $\Delta Q/\Delta T$, and take the limit as $\Delta T$ goes to zero, assuming that this limit exists, and obtain the *heat capacity* $\delta Q/\delta T$ of the given mass of substance for this particular process. For liquids and solids this limit is found to be practically independent of stress distribution, as well as the method of taking the limit, and to vary slowly with temperature, whence the limit is essentially unique. The limit $\delta Q/\delta T$ is not to be confused with a derivative $\partial Q/\partial T$ of a function $Q$. In general $Q$ is not a function of the variables $p$, $v$, and $T$, but depends on the process used. The limit $\delta Q/\delta T$ per unit mass of a body is called the *specific heat* of the body. In engineering tables the specific heat is often given as the quotient of the limit $\delta Q/\delta t$ for a unit mass of the given liquid or solid to the limit $\delta Q/\delta t$ for a unit mass of water, where the water is at a specified temperature and pressure. Later we shall use specific heat as the limit $\delta Q/\delta t$ per unit weight of the body.

Typical specific heats in the sense mentioned above for engineering tables are 0.3 for coal, 0.5 for gasoline, 0.6 for alcohol, 0.15 for glass, 0.22 for brick, 0.1 for iron, and 0.033 for mercury.

For water at 1 atmosphere and 32 degrees Fahrenheit the specific heat $c_p$ is 1.001, whereas at the same pressure but 212 degrees Fahrenheit this constant is 1.021. At 200 degrees Fahrenheit and 1,000 pounds per square inch pressure the constant $c_p$ for water is 1.0, whereas at the same temperature but 6,000 pounds per square inch we have $c_p = 0.98$. For iron at 600 degrees Fahrenheit we have $c_p = 0.127$, whereas at 2,500 degrees Fahrenheit we have $c_p = 0.167$.

For gases it is convenient to use two specific heats, the $c_p$ being $\delta Q/\delta T$ at constant pressure, and the other the specific heat $c_v$ being $\delta Q/\delta T$ at constant volume, these limits being independent of the method of taking the limit. In engineering these specific heats are given in British thermal units per pound per degree Fahrenheit. For air at 90 degrees Fahrenheit the specific heat $c_p$ increases from 0.24 at zero pressure to 0.27 at 100 atmospheres, while for this pressure range $c_v = 0.17$. At 6,000 degrees

Fahrenheit and the same pressure range we have $c_p = 0.31$, $c_v = 0.24$. This gives an idea of the variation of these specific heats with pressure and temperature.

It is erroneous to think of a body at a given temperature as containing a corresponding amount of heat. For example, we can show experimentally, on the basis of the laws for gases, that in bringing air from 0 degrees centigrade and 1 atmosphere to 100 degrees centigrade and 2 atmospheres, the heat required depends on the process used. Thus the air can be brought to 100 degrees centigrade by heating it under constant pressure, and to 2 atmospheres by then compressing it at constant temperature. On the other hand, we might compress the air to 2 atmospheres at constant temperature, and then heat it under constant pressure to 100 degrees centigrade. The number of British thermal units absorbed is different for each process.

Heating a substance, by which we mean adding heat to the substance, is not always accompanied by a corresponding change in temperature. In the ordinary transition from one of the liquid, gaseous, or solid phases of a substance to another of these phases there is a transfer of heat, called *latent heat*, while temperature and pressure remain constant. Thus heat is required to melt ice, and to vaporize water.

**6.5. Internal energy.** In this section we shall not be concerned with energy possessed by a body due to its position, or motion relative to other bodies, that is, we shall not study kinetic and potential energies of the body as a whole. The states here mentioned will refer to states of thermal equilibrium. If a system goes from a state $A$ to a state $B$ without any heat transfer between it and its environment, we say that the change of state has occurred *adiabatically*. The following statement is one form of the first law of thermodynamics.

***Law* 6.2.** *If a system goes from one state to another adiabatically, the work done is independent of the manner in which the system went from the one state to the other.*

Law 6.2 is an experimental one. There are countless ways of going from the first state to the second adiabatically through sequences of intermediate states, but in each case the work involved is the same, thus depending only on the initial and final states. We can then associate with each state $X$ a real number, which is the work $U$ necessary to bring the system from a fixed basic state of reference adiabatically to the state $X$. Thus the work necessary to bring the system adiabatically from a state $A$ to a state $B$ is the work $U_B$ necessary to bring the system from the basic state $C$ adiabatically to the state $B$, minus the work $U_A$ necessary to bring the system from the basic state $C$ adiabatically to the state $A$, that is, $(U_B - U_A)$. We call $U$ the *internal energy* of the system.

A substance thermally insulated from its environment can be made to go from one state to another without doing mechanical work on the substance, as when an electric resistor is immersed in the substance. By passing an electric current through the resistor the substance may be heated.

By experiment we know that it is not always possible to take a system from any state $A$ to any state $B$ adiabatically. We can still get the difference in the internal energy function for the two states.

Suppose that we go from state $A$ to state $B$ adiabatically. The corresponding work is called *adiabatic work.* If, instead, we measure the amount of work in changing from $A$ to $B$ nonadiabatically, we find from experiment that this is not the adiabatic work $(U_B - U_A)$. For the principle of the conservation of energy to be valid, as demanded by experience, we must assume that an energy transfer has taken place between the given system and its surroundings due to the temperature difference between the system and its surroundings, which energy we have introduced above as heat. We have now the following form of the first law of thermodynamics.

*Law* **6.3. (First law of thermodynamics).** *If a system goes from a state $A$ with internal energy $U_A$ to a state $B$ with internal energy $U_B$, and $Q$ is the heat transferred between the system and its surroundings, and further $W$ is the work done on or by the system during the process, the relation*

$$Q = U_B - U_A + W \tag{6.7}$$

*holds.*

It is convenient to have $U_B > U_A$ when $Q = 0$ and work is done on the system. Relation (6.7) then shows that $W$ is negative. In this chapter we shall therefore follow the convention that the work $W$ is positive when the system does work, and negative in the contrary case. If we have a process with no work ($W = 0$), equation (6.7) shows that $Q$ is positive when $U_B > U_A$. It is therefore convenient to say that $Q$ is positive when heat is added to the system, and negative otherwise.

It is clear that changes which occur in a region bounded by an insulating wall, like the wall of a refrigerator, are adiabatic for practical purposes. There are numerous other places where the engineer may assume that a process occurs adiabatically without the presence of material usually considered to be insulating. Thus if the changes which a gas undergoes are rapid enough, there is little time for the heat to travel through the boundary material. Then for practical purposes, where refined calculations are not necessary, the process is adiabatic. This is true when gases pass through the wheel of a gas turbine, when a liquid squirts through a nozzle, and when the gasoline mixture in an internal combustion engine is compressed. Even though a process is not adiabatic, it is often wise to treat it as if it were, and make corrections as necessary.

The first law of thermodynamics is a special case of the following principle in the sense that this law involves only certain forms of energy.

*The principle of the conservation of energy is that energy can neither be created nor destroyed. Our definitions of the various forms of energy are made in such a way that this principle holds;* in fact, we fundamentally refer all energy to the mechanical energy of doing work.

Suppose that, in contradiction to Law 6.2, there were two adiabatic processes for bringing a system from a state $A$ to a state $B$ with different amounts of work. Let $W_1$ be the work for the first process and $W_2$ the work for the second. If we could reverse the second process the work involved would be $-W_2$. The net work in going from $A$ to $B$ by the first process and from $B$ to $A$ by the reverse of the second process is then $(W_1 - W_2)$. We can repeat this cycle $n$ times with the corresponding work $n(W_1 - W_2)$. Suppose that $(W_1 - W_2)$ is positive. We can then obtain any amount of energy in the form of mechanical work from the given system, so that we would have a case of *perpetual motion.* The assumption of the validity of Law 6.2 precludes this possibility.

**6.6. Work.** A substance under a uniform pressure $p$ is compressed by a piston in a cylinder. The cross-sectional area of the cylinder is denoted by $A$, so that the force on the piston is $pA$. Suppose that the piston is moved through a distance $\Delta x$. For this movement the pressure $p$ varies between a minimum value $p_m$ and a maximum value $p_M$. It follows that for some properly chosen value $\bar{p}$ between $p_m$ and $p_M$ the work $\Delta W$ done on the substance is $\bar{p}A\Delta x$. Let $V$ be the volume of the cylinder. Since $\Delta V = A\Delta x$, we can write

$$\Delta W = \bar{p}dV \tag{6.8}$$

when $V$ is taken as an independent variable.

Suppose now that a substance under a uniform pressure $p$ is compressed, but not necessarily by a piston in a cylinder. We can imagine that the substance is immersed in an incompressible fluid contained in a cylinder. When the piston displaces a volume $\Delta V$, the substance is compressed by this amount. The work done on the substance is then the same as the work done on the fluid by the piston. Thus (6.8) holds in any case.

It is readily seen that (6.8) is valid for expansion of the substance as well as compression. Substituting (6.8) into (6.7), replacing $Q$ by $\Delta Q$, and $(U_B - U_A)$ by $\Delta U$ we have

$$\Delta Q = \Delta U + \bar{p}dV. \tag{6.9}$$

Replace $\bar{p}$ by the initial pressure $p$. Since $U$ is a function of the thermodynamic variables (two of the quantities $p$, $V$, and $T$), it is reasonable to suppose that $U$ has a total differential, so that (6.9) can be replaced by the relation

$$\delta Q = dU + pdV, \tag{6.10}$$

which defines a quantity $\delta Q$, where we assume that $\delta Q$ differs from $\Delta Q$ by an $\epsilon$-term which can be neglected. This $\epsilon$-term does not contribute to the final limits in processes where we shall let $\Delta Q$, $\Delta U$, and $\Delta V$ go to zero. Formula (6.10) is called the *differential form of the first law of thermodynamics.* In integrating to obtain the total heat $Q$ for a process in which we go from a state $A$ to a state $B$, formula (6.10) yields precise results for the limit that defines the integral.

**6.7. Enthalpy.** The energy function $(U + pV)$, which often occurs in thermodynamic studies (see Section 6.8), is denoted by $H$ and is called *enthalpy.* We suppose that $H$ has a total differential $dH$ as required for what follows. In engineering practice enthalpy is rather generally given in British thermal units per pound. In this case we use the symbol $h$ for enthalpy, and write $h = u + pv$, where $u$ and $v$ are respectively the internal energy and volume per pound. If $p$ is in pounds per square foot and $V$ is in cubic feet, the quantity $pV$ has the dimensionality of foot-pounds, and therefore can be interpreted as energy. For flowing fluids the *flow work,* or *work of displacement* is the difference between values of $pV$. From (6.10) it follows that

$$dH = \delta Q + Vdp. \tag{6.11}$$

For an isobaric process this gives

$$dH = \delta Q$$

so that the heat involved in going from a state $A$ to a state $B$ at constant pressure is equal to the change in enthalpy. In the past enthalpy has often been referred to as *heat content.* From the definition of specific heat we have

$$H_B - H_A = \int_A^B c_p dT$$

for the enthalpy at $B$ minus the enthalpy at $A$, the integration being over the range of intermediate states.

Consider an adiabatic process from $A$ to $B$. We have $\delta Q = 0$ for the transition from any one state in the process to any other. By (6.11)

$$H_B - H_A = \int_A^B Vdp.$$

the integration being over the process.

**6.8. Enthalpy in applications.** In the part of the book on fluid mechanics we shall neglect the internal energy changes, heating, and external work done on the fluid. We shall not neglect these quantities in the present section. Let us now consider flow, such as steady flow (see Chapter 12), for which an energy density balance equation can be written relating the energy densities of the fluid at two positions occupied by a particle (or simply point) $P$ where $P$ moves from the one position to the other with the fluid. The reader is not to associate mass with this particle. The word "particle" is used here merely to designate a point which moves with the fluid. In the initial state the particle $P$ is at an elevation $z_1$ with respect to some reference plane, and at $z_2$ in the final state. We let $V_{s1}$ be the speed of $P$ at the start and $V_{s2}$ the speed at the finish. Similarly, $p_1$, $v_1$, and $u_1$ are, respectively, the initial pressure, specific volume (volume per unit weight in this case), and internal energy density (per unit weight) at $P$, and $p_2$, $v_2$, and $u_2$ the final values of these quantities.

For the moment consider a cylinder with cross-sectional area $A$, and cross sections labeled 1 and 2 respectively (see Figure 125). Suppose that fluid is flowing along the cylinder in the positive direction of the $x$-axis, drawn parallel to the cylinder in the figure. We let $I$ denote a cross section of the cylinder between 1 and 2, and let $S$ be the substance between sections 1 and $I$ at a given instant. We suppose that at a later instant the left face of the substance $S$ is at the section $I$, while the right face is at a section $II$, and that still later the left face of $S$ is at $II$, while the right face is at a section $III$. Continuing this process, the substance $S$ is eventually between sections $N - 1$ and $N$ for some value of $N$, where the section $N$ is to the right of section 2. We let $E_I$ denote the volume element between sections 1 and $I$, $E_{II}$ the volume element between sections $I$ and $II$, etc. The width of the element $E_i$ will be denoted by $\Delta x_i$ for $i = I, II, \ldots, N$. We let $p$ and $v$ be the pressure and specific volume, assumed to be uniform over a cross section of the cylinder. When $S$ moves from $E_I$ to $E_{II}$, the fluid to the left of $S$ does the work $p_I A \Delta x_I$ on $S$, where $p_I$ is the value of $p$ at some properly chosen cross section of $E_I$. The substance $S$ meanwhile does the work $p_{II} A \Delta x_{II}$ on the fluid to the right, where $p_{II}$ is the value of $p$ at some cross section of $E_{II}$. The net work done on $S$ in going from $E_I$ to $E_{II}$ is thus $p_I A \Delta x_I - p_{II} A \Delta x_{II}$. Similarly, the net work in moving $S$ from $E_{II}$ to $E_{III}$ is $p_{II} A \Delta x_{II} - p_{III} A \Delta x_{III}$. The net work in moving $S$ from $E_I$ to $E_N$ is thus a sum

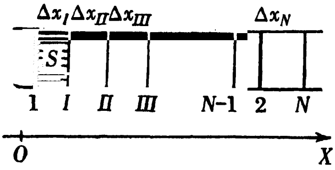


**Fig. 125. Substance $S$ flowing from section 1 to section 2.**

$$(p_I A \Delta x_I - p_{II} A \Delta x_{II}) + (p_{II} A \Delta x_{II} - p_{III} A \Delta x_{III}) + \cdots + (p_{N-1} A \Delta x_{N-1} - p_N A \Delta x_N),$$

or simply

$$p_I A \Delta x_I - p_N A \Delta x_N.$$

We let $\Delta w$ be the weight of $S$. The quantity $A\Delta x_I$ is then $v_I\Delta w$ for the specific volume $v_I$ evaluated at some properly chosen section of $E_I$, whereas $A\Delta x_N$ is $v_N\Delta w$ for a value $v_N$ of $v$ at a cross section of $E_N$. The work done in moving $S$ from $E_I$ to $E_N$ per unit weight is now

$$(p_Iv_I - p_Nv_N).$$

Taking the limit at $\Delta x_I$ goes to zero this work becomes (assuming that the limit exists) $p_1v_1 - p_2v_2$, where $p_1$, $v_1$ are the values of $p$ and $v$ at the section 1, and $p_2$, $v_2$ are the values at section 2. This work is called the ***flow work*** associated with sections 1 and 2, although it is actually the density of this work associated with these sections. *If there is no motion of the fluid relative to the observer there is no flow work.*

Since the cylinder may be arbitrarily small, the flow work $p_1v_1 - p_2v_2$ can be taken to be the density of this work associated with two points of the substance, rather than for cross sections of the cylinder. Further, the cylinder may be a purely geometric one placed in a moving fluid with elements parallel or approximately parallel to the direction of flow of the fluid at points of the fluid at which the flow work is treated. By mathematical limit methods flow work $p_1v_1 - p_2v_2$ is associated with each pair of points $P_1$ and $P_2$ occupied by a fluid particle $P$ when $P$ moves under flow of the fluid from the one point to the other.

Let $m$ be the mass of a body with weight $w$ and traveling with a speed $V_s$. The kinetic energy of the body is $\frac{1}{2}mV_s^2$, but since $w = mg$, this energy is $wV_s^2/2g$. It follows that the kinetic energy density is $V_s^2/2g$ per unit weight of the body.

In the ordinary flow of fluids we are concerned with energy changes which involve only heat, internal energy, potential energy, kinetic energy, work, and energy losses such as those due to friction. The potential energy is generally due to gravity. We measure this energy in terms of a vertical coordinate $z$. Returning to the particle $P$ that moves with the fluid, we note that the energy density at $P_1$ is

$$u_1 + \frac{V_{s1}^2}{2g} + z_1.$$

There is a corresponding energy density at the final state with the subscript 1 replaced by 2, and flow work $p_1v_1 - p_2v_2$ is associated with these states. Between the initial and final states external work may be done on or by the fluid at and near $P$, so that we may have to associate a corresponding work density $W$ with the movement of $P$ from the initial state to the final state. To take care of *energy losses* due to *friction*, or other forms of energy such as chemical energy, we associate an energy density $K$ with the movement of $P$ from its initial state to its final state. We have the energy density balance equation

$$Q = (u_2 - u_1) + (p_2v_2 - p_1v_1) + (z_2 - z_1) + \frac{(V_{s2}^2 - V_{s1}^2)}{2g} + W + K, \quad (6.12)$$

where $Q$ is the heat density associated with the movement of the particle $P$ from the initial state to the final state. Equation (6.12) is the general energy equation for the flow of fluids. In deriving this equation we have assumed that we have given some frame of reference with respect to which we measure the velocity of the fluid. This frame of reference might be a pipe through which the fluid is flowing.

In applying (6.12) it is obviously necessary to measure the energy in the same units. Thus if we use British thermal units per pound for $Q$, the $u$'s and $W$, and we use feet for the $pv$'s, $z$'s, and $(V_s^2/2g)$'s we must divide the terms in the latter by $J$, where $J$ is the number of foot-pounds per British thermal unit, that is, 778. The symbol $J$ is *Joule's constant.*

Equation (6.12) is not the only energy density equation needed in the flow of fluids. It is generally not sufficient to treat energy density only relative to a frame of reference with respect to which the fluid is moving. We need also an equation relating energy densities relative to a frame of reference at the point at which the energy density is taken. Therefore for the initial state of the fluid we consider a sphere $S_1$ of radius $\Delta r$ with center at the given particle $P$. This sphere bounds a region $R_1$ of the fluid, and in the final state the sphere $S_1$ has gone into a surface $S_2$ bounding a region $R_2$. We set up a Cartesian $xyz$-coordinate system, assuming that we can somehow identify points $M$ and $N$ in $R_1$, so that $P$, $M$, and $N$ are distinct and not on the same line. We can take the origin of the coordinate system at $P$, the $x$-axis through the point $M$, and the $xy$-plane through the point $N$, whence a coordinate system is determined, provided positive directions are taken for the axes. We suppose that the points $M$ and $N$ are so chosen that in the final state $P$, $M$, and $N$ do not lie on the same line, and that they still determine an $xyz$-coordinate system. In going from the initial state to the final state the sphere $S_1$ is assumed to be continuously deformed into the surface $S_2$, and $R_1$ is continuously deformed into the region $R_2$ in the topological sense.

Let $\rho$ be the density of the fluid at a point in the region $R_1$. We assume that $\rho$ is a continuous function of $x$, $y$, and $z$. Suppose that the $xy$-plane attached to $P$, $M$, and $N$ is horizontal. Relative to the $xy$-plane the fluid in $R_1$ has the potential energy

$$g \iiint\limits_{R_1} \rho z dV.$$

By the first theorem of the mean of the integral calculus this energy is $g\rho_1 z_1 V_1$ for the volume $V_1$ of the region $R_1$, and a properly chosen value $\rho_1 z_1$ of $\rho z$ between the minimum and maximum taken on at the points of $R_1$. The mass of the fluid in the region $R_1$ is $\rho^* V_1$ for a properly chosen value $\rho^*$ of $\rho$ for this region. Thus the potential energy density per pound relative to the $xy$-plane is

$$\frac{\displaystyle\iiint_{R_1} \rho z dV}{\displaystyle\iiint_{R_1} \rho dV} = z_1\left(\frac{\rho_1}{\rho^*}\right).$$

Since $\rho$ is continuous at $P$ the limit of the fraction $\rho_1/\rho^*$ is 1 as $\Delta r$ goes to zero. Since $z_1$ goes to zero with $\Delta r$, the potential energy density at $P$ is zero, it being understood that the $xy$-plane is the reference plane. It follows that the potential energy density at $P$ relative to a horizontal (perpendicular to the force of gravity) plane through $P$ is zero.

Letting primes denote derivatives with respect to time, the kinetic energy of the fluid in the region $R_1$ is

$$\tfrac{1}{2}\iiint_{R_1} \rho(x'^2 + y'^2 + z'^2)dV,$$

by definition. We assume that the integrand function is continuous over the region $R_1$, so that

$$\tfrac{1}{2}\iiint_{R_1} \rho(x'^2 + y'^2 + z'^2)dV = \tfrac{1}{2}\rho_1^*(x'^2 + y'^2 + z'^2)^* V_1$$

for a value $\rho_1^*(x'^2 + y'^2 + z'^2)^*$ between the minimum and maximum taken on by the integrand function over the region $R_1$. Thus the average kinetic energy density over the region $R_1$ is

$$\frac{1}{2}(x'^2 + y'^2 + z'^2)^*\left(\frac{\rho_1^*}{\rho^*}\right).$$

The kinetic energy density at $P$ is

$$\tfrac{1}{2}\lim_{\Delta r \to 0}(x'^2 + y'^2 + z'^2)^*.$$

Consider a particle $G$ of the region $R_1$, where $G$ has the coordinates $(\Delta x, \Delta y, \Delta z)$ relative to the coordinate system attached to the points $P$, $M$, and $N$. We let $u$, $v$, and $w$ represent $x$-, $y$-, and $z$-velocity components of a particle relative to this coordinate system. Consider the relations

$$\begin{aligned}
\Delta u &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z,\\
\Delta v &= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \frac{\partial v}{\partial z}\Delta z,\\
\Delta w &= \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z,
\end{aligned}$$

where the partial derivatives are evaluated at the point $P$, and we are neglecting $\epsilon$-terms (terms which do not contribute to the final limits). The components $u$, $v$, and $w$ are also functions of time $t$, but since we are considering the initial state only, the time does not enter into the present argument. The $x$-, $y$-, and $z$-velocity components of $G$ are given by $u + \Delta u$, $v + \Delta v$, and $w + \Delta w$ respectively, where $u$, $v$, and $w$ represent the velocity components at the point $P$. Since $u$, $v$, and $w$ vanish, the velocity components of $G$ are given by $\Delta u$, $\Delta v$, and $\Delta w$. The partial derivatives above will be assumed to be bounded in absolute value by a number $\lambda$, whence

$$(\Delta u)^2 + (\Delta v)^2 + (\Delta w)^2 \leqq 3\lambda^2[|\Delta x| + |\Delta y| + |\Delta z|]^2 = 27\lambda^2(\Delta r)^2.$$

It follows that the limit of the maximum value of $[(\Delta u)^2 + (\Delta v)^2 + (\Delta w)^2]$ goes to zero with $\Delta r$. Thus the kinetic energy density at $P$ is zero.

Since the particle $P$ is stationary with respect to the $xyz$-coordinate system attached to it, there is no flow work density at $P$.

Let $W^*$ be the work density associated with the point $P$ as it goes from the initial state to the final state, where this work is relative to the $xyz$-coordinate system attached to $P$.

In place of (6.12) we now have

$$Q = u_2 - u_1 + W^* + K^* \qquad (6.13)$$

for a term $K^*$ that takes care of energy losses and other forms of energy such as chemical energy if such must be taken into account. For points of the fluid in a region near $P$, formula (6.13) is valid for practical purposes. Equation (6.13) is a generalized form of (6.7). *We have shown that* (6.7), *or its generalized form* (6.13), *holds for fluids in motion* as well as at rest. Both (6.12) and (6.13) are needed to study a given fluid motion thermodynamically. An application of the differential form (6.10) of (6.7) will be found in Section 6.9, where this equation is employed in treating adiabatic processes which may involve the flow of gases.

We can simplify the formula (6.12) by the use of enthalpy, letting $h_1$ and $h_2$ be the initial and final enthalpies. We then have

$$Q = (h_2 - h_1) + (z_2 - z_1) + \frac{(V_{s2}^2 - V_{s1}^2)}{2g} + W + K. \qquad (6.14)$$

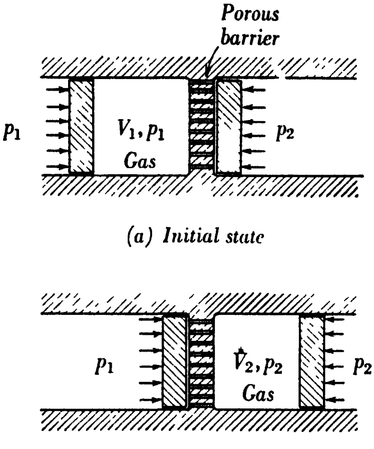
In engine problems the change in potential energy as the fluid passes through the engine is generally negligibly small so that we may take $z_1 = z_2$ when studying states just before, just after, and in the engine.

For the flow of a fluid through a nozzle the process is approximately adiabatic with no external work and energy losses so that (6.14) becomes

$$h_1 - h_2 = \frac{(V_{s2}^2 - V_{s1}^2)}{2g}. \qquad (6.15)$$

In ordinary applications law (6.15) may be assumed to be satisfied to 90 per cent or 95 per cent efficiency in the sense that efficiency is the ratio $(V_2^2/2g)/[h_1 - h_2 + (V_1^2/2g)]$.

In many engineering problems, as in the case of flow of steam through a turbine, good accuracy can be obtained by taking the *friction loss K proportional to the enthalpy difference* where this difference is computed on the basis of an adiabatic process.



(a) *Initial state*

(b) *Final state*

**Fig. 126. Throttle.**

For flow through a *reaction turbine* the kinetic energy change is small (by definition). Since the flow is approximately adiabatic, equation (6.14) yields

$$h_1 - h_2 = W, \tag{6.16}$$

where we have taken $K = 0$. Due to energy losses we actually have

$$\eta(h_1 - h_2) = W \tag{6.17}$$

in place of (6.16) where $\eta$ is the *efficiency of the turbine*. Relation (6.16) will often check to 70 per cent efficiency. Actual efficiency values are obtained from experiment.

For a *throttle* we have simply

$$h_1 = h_2 \tag{6.18}$$

as may be seen as follows. By definition a throttle performs a throttling process as illustrated in Figure 126. Gas at a constant pressure $p_1$ is forced through a porous barrier to a region beyond the barrier where the pressure is kept at a value $p_2$. The porous barrier permits the gas to flow through the barrier but isolates the pressures on both sides of the barrier. The initial volume of the gas is taken to be $V_1$, and the volume after the gas has passed through the barrier is $V_2$. The work done on the gas to the left of the barrier in Figure 126 is then $p_1V_1$, and the work done by the gas on the right is $p_2V_2$, so that the net work is $(p_2V_2 - p_1V_1)$. The system is assumed to be thermally insulated during the throttling process, so that the amount $Q$ of heat transferred to or from the surroundings is zero. By (6.12) with $K = 0$ we have

$$0 = U_2 - U_1 + p_2V_2 - p_1V_1$$

for the initial and final internal energies $U_1$ and $U_2$ of the gas, whence (6.18) follows.

In the case of a practical throttle we can apply (6.18) provided that the gas is traveling slowly, if not in the immediate neighborhood of the throttle then at some distance on either side of the throttle.

We often have occasion to work with the enthalpy of a mixture of substances chemically inert in the mixture. Suppose that we have substances $A$ and $B$, where these substances are at the same pressure $p$ and temperature $T$. Because it is customary to work with enthalpy per pound, we shall treat the weights $w_1$ and $w_2$ of $A$ and $B$, respectively, instead of their masses. We let $v_1$ be the specific volume of $A$, and $v_2$ the specific volume of $B$. For convenience the reader may imagine that these substances are in a chamber separated by a barrier $C$ of negligible thickness. If now the barrier $C$ is removed, the system will still be at the pressure $p$ and temperature $T$, the substances $A$ and $B$ being in thermal equilibrium at the start. We know that even though the substances become thoroughly mixed due to molecular movement, the pressure and temperature will remain the same, at least we treat only such mixtures. We can imagine that if we could separate the molecules of $A$ and $B$, and replace the barrier, the substances $A$ and $B$ would still be at the pressure $p$ and temperature $T$.

We let $u_1$ be the internal energy of $A$ per unit weight, and $u_2$ the corresponding internal energy of $B$. By the conservation of energy, and the fact that work and heat exchanges are not involved, the internal energy $U$ of the mixture is due to the internal energies of $A$ and $B$, and per unit weight of the mixture the internal energy $u$ is given by

$$u = \frac{w_1u_1 + w_2u_2}{w_1 + w_2}.$$

The initial volume occupied by $A$ is $w_1v_1$, and the volume so occupied by $B$ is $w_2v_2$. The specific volume $v$ of the mixture is then

$$v = \frac{w_1v_1 + w_2v_2}{w_1 + w_2}.$$

From the definition of enthalpy, the enthalpy $h$ of the mixture is given by

$$h = u + pv.$$

From the formulas for $u$ and $v$ we can write

$$h = \frac{w_1 h_1 + w_2 h_2}{w_1 + w_2}$$

for the initial enthalpies $h_1$ and $h_2$ of $A$ and $B$. The enthalpy of a mixture thus comes from the enthalpies of its components as shown. The total *enthalpic energy* $h(w_1 + w_2)$ *of the mixture is the sum* $(w_1h_1 + w_2h_2)$ *of the total enthalpic energies* $w_1h_1$, $w_2h_2$ *of the components.* This obviously extends to a mixture of any number of components.

We shall now concern ourselves with the mixture of $r$ perfect gases. It is an experimental fact that vapors and gases when mixed behave as if each constituent fills the entire volume. This property is assumed to hold for perfect gases. Let there be $n_1$ moles of a perfect gas, $n_2$ moles of another, etc. Suppose that we have $r$ such gases thoroughly mixed. We introduce the partial pressures $p_1, p_2, \ldots, p_r$, where

$$p_1 = \frac{n_1}{V} RT, \; p_2 = \frac{n_2}{V} RT, \; \ldots, \; p_r = \frac{n_r}{V} RT$$

for the volume $V$ occupied by the mixture, the absolute temperature $T$ of the mixture, and the gas constant $R$. If $p$ is the pressure of the mixture we know by equation (6.6) that

$$p = p_1 + p_2 + \cdots + p_r.$$

Let $v_1$ be the specific volume of the first gas when this gas fills the entire volume, and $u_1$ the corresponding internal energy density. The enthalpy $h_1$ of this gas is then

$$h_1 = u_1 + p_1 v_1,$$

and the total enthalpic energy $H_1$ of this gas is given by

$$H_1 = U_1 + p_1 V$$

for the total internal energy $U_1$. For the second gas we have

$$H_2 = U_2 + p_2 V.$$

Thus for the $r$ gases we have

$$H_1 + \cdots + H_r = (U_1 + \cdots + U_r) + (p_1 + \cdots + p_r)V.$$

*The sum of the internal energies* $U_1, \ldots U_r$ *of relatively inert perfect gases is the internal energy* $U$ *of the mixture.* It follows that the above relation can be written as

$$H_1 + \cdots + H_r = U + pV. \tag{6.19}$$

From the definition of enthalpy the relation (6.19) shows that the total enthalpic energy $H$ of the given mixture of $r$ gases is the sum of the enthalpic energies $H_1, \ldots, H_r$ of these gases.

Thus the *total enthalpic energy of a mixture of relatively inert perfect gases is the sum of the total enthalpic energies of the constituent gases, where these energies are computed on the basis that each gas fills the entire volume.*

Consider a combustion process in which heat energy is liberated. This energy is called the *heat of combustion,* and comes from the chemical energy in the constituents.

Suppose that the combustion process takes place adiabatically. The heat of combustion may be defined to be the quantity of heat rejected when the products of combustion are brought back to the temperature and pressure of the mixture before the reaction. By (6.11) the heat of combustion equals the change in enthalpy.

When a substance changes phase, *latent heat* is given up by the substance, or taken on. If latent heat is given up this must come from the enthalpic energy of the substance, and there will be a corresponding change in enthalpy. Changes of phase for a *pure substance* (not a mixture) occur at constant pressure and temperature. However, the specific volume $v$ changes with the phase, and this implies a corresponding change in the enthalpy of the substance. Since the pressure $p$ is constant we have $dp = 0$, whence (6.11) implies that the latent heat is equal to the change in enthalpy.

When the enthalpy changes because of variations in both $u$ and $pv$, it is often convenient to consider first the enthalpy change due to a change in $u$, and then compute the additional enthalpy change due to the change in $pv$. The change in $u$ is often large compared to the change in $pv$, so that for practical purposes we may in such cases take the change in the enthalpy $h$ to be the change in the internal energy $u$.

### 6.9. Adiabatic, isothermal, and isobaric processes.

Experiments show that *the internal energy of a perfect gas is a function of the temperature only.* We therefore write

$$u = f(T),$$

where $u$ is the internal energy of one mole of the gas. Relation (6.10) now becomes

$$\delta Q = f'(T)dT + pdv \tag{6.20}$$

for the volume $v$ occupied by one mole of the gas. In Section 6.4 we gave definitions for the specific heats $c_v$ and $c_p$. These quantities are rigorously defined by (the following derivatives are assumed to exist)

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v, \quad c_p = \left(\frac{\partial h}{\partial T}\right)_p,$$

where $v$ is constant in $\partial u/\partial T$, and $p$ is constant in $\partial h/\partial T$, while $h$ is the enthalpy (per mole in the present section). Thus

$$c_v = f'(T)$$

so that (6.20) becomes

$$\delta Q = c_v dT + p dv \tag{6.21}$$

for the specific heat $c_v$ at constant volume. From the general gas law

$$p dv + v dp = R dT. \tag{6.22}$$

Combination of (6.21) and (6.22) gives

$$\delta Q = (c_v + R) dT - v dp.$$

It follows that for perfect gases

$$c_p = c_v + R$$

and

$$\delta Q = c_p dT - v dp. \tag{6.23}$$

In an *adiabatic process* $\delta Q = 0$. From (6.21) and (6.23) we have

$$\frac{dp}{p} = -\gamma \frac{dv}{v}, \tag{6.24}$$

where $\gamma$ is the quotient $c_p/c_v$ of the specific heats. For practical purposes when the range of $\gamma$ is not too large we may treat $\gamma$ as if it were constant. In fact, for air in the range 200 degrees Rankine to 6,400 degrees Rankine and pressures under 100 atmospheres the quantity $\gamma$ varies only from 1.401 to 1.280, respectively. Integrating (6.24) we have

$$p v^{\gamma} = K \tag{6.25}$$

for a constant $K$ depending on the process. Relation (6.25) is based on the assumption that all states considered are states of thermal equilibrium.

The constant $K$ in (6.25) is not frequently used directly in computations. Rather we let $p_1$, $v_1$ be values of $p$ and $v$ for an initial state of a gas, and $p_2$, $v_2$ values of these variables for the final state of the gas. It is assumed that the gas goes from the initial to the final state adiabatically. By (6.25) we have

$$\frac{p_1}{p_2} = \left(\frac{v_2}{v_1}\right)^{\gamma},$$

and by the general gas law with $T_1$ and $T_2$ as the temperatures for the initial and final states, respectively, we have

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}, \tag{6.26}$$

which is in common use, and on the basis of which many tables have been constructed. Thus, given the initial temperature $T_1$, pressure $p_1$, and the final pressure $p_2$, we can compute the final temperature $T_2$.

As will be seen from examples, various nonadiabatic processes are important for engineering purposes. When the temperature is constant (and

Boyle's law applies) we have an *isothermal* process. When the pressure is constant the process is *isobaric* and $dp = 0$. Some changes occur at constant volume whence $dv = 0$.

From the definition of *enthalpy* $h$, we find that *for a perfect gas*

$$dh = c_v dT + R dT = c_p dT.$$

We have

$$\delta Q - \delta Q'' = du + p dv = dh - v dp,$$

where $\delta Q''$ is the energy lost through friction. For an adiabatic process $\delta Q = 0$. Writing $\delta Q'' = \zeta dh$, we have

$$(1 + \zeta) dh = v dp$$

for such a process. Thus

$$(1 + \zeta) c_p dT = v dp.$$

Since $pv = RT$, we have $dT = (p dv + v dp)/R$, so that substitution for $dT$ in the last equation yields

$$\frac{dp}{p} = -\frac{(1 + \zeta) c_p dv}{[(1 + \zeta) c_p - R] v}.$$

Thus $\gamma$ in equation (6.24) is replaced by the quantity $n$, where

$$n = \frac{(1 + \zeta) c_p}{[(1 + \zeta) c_p - R]} = \gamma \frac{1 + \zeta}{1 + \zeta\gamma}$$

in view of $c_p = c_v + R$. Thus the adiabatic law (6.25) is replaced by

$$pv^n = K. \tag{6.27}$$

A process for which (6.27) holds is called *polytropic*. In such calculations as the computation of states for the flow of steam through a turbine or the flow of air through a compressor or turbine wheel of a gas turbine, it is often advisable for the attainment of high accuracy to use a value of $\zeta$ based on experience, and then to calculate the states on the basis of the corresponding law (6.27).

Suppose that at the end of a process the initial state of the given physical system under study and the surroundings may be restored. Such a process is termed *reversible*. There are actually no physical reversible processes, that is, all processes in nature (except possibly living processes) are irreversible, as follows from the second law of thermodynamics to be given later. There are many processes which, however, may be treated as reversible for practical purposes. Thus when a gas in a cylinder is allowed to expand slowly against a well-oiled spring-loaded piston, the gas being thermally insulated from the surroundings by the material of the piston and the cylinder, the process may be considered adiabatic and reversible. Here we have made such obvious and reasonable assumptions as that there is no leakage of the gas.

A reversible adiabatic process is called *isentropic.* The word *entropy* will be defined later. The compression of air by the compressor of a gas turbine is an example of a process often considered to be isentropic, at least when the utmost accuracy is not required.

Suppose that a gas in a cylinder expands from a volume $V_1$ to a volume $V_2$ because of the instantaneous movement of the piston in the cylinder from one position to another. To reduce the volume of the gas back to $V_1$ it is necessary to exert external work on the gas, whereas the gas does no work in expanding. This is an example of an irreversible process.

The concept of reversibility will be needed later when we wish to reverse paths of integration on the $pv$-plane where these paths join two given states.

**6.10. Enthalpy of gases.** For any process the enthalpy difference $\Delta h$ in the case of a perfect gas satisfies the relation

$$\Delta h = \int_1^2 c_p dT, \tag{6.28}$$

where the integration is from the state 1 to the state 2. For an *isothermal process* the enthalpy $h$ is clearly a constant.

Let $p_1$ and $p_2$ be the initial and final pressures of a perfect gas, and let $T_1$ and $T_2$ be the initial and final temperatures respectively. For an isentropic process the enthalpy difference $\Delta h$ can be computed from $p_1$, $p_2$, and $T_1$ by the use of (6.26) and (6.28).

**6.11. Flow of gas through a nozzle.** What follows will apply to convergent nozzles. A *convergent nozzle* is one whose cross section narrows as the opening in the direction of flow is approached. We shall first consider the flow of a perfect gas through the nozzle. We let $p_1$, $T_1$, $v_1$, and $h_1$ be, respectively, the pressure, temperature, specific volume (per pound), and enthalpy ahead of the nozzle, and $p_2$, $T_2$, $v_2$, and $h_2$ the corresponding values at the nozzle orifice. We shall suppose that the velocity of the gas ahead of the nozzle can be neglected in what follows. Let $V_{s2}$ be the speed of the gas at the nozzle orifice, and $A$ the area of the cross section of the nozzle at the orifice. The symbols $R^*$ and $g$ denote the gas constant and magnitude of the acceleration of gravity as usual. We assume adiabatic flow so that

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

for the adiabatic exponent $\gamma$. Since $dh = vdp$

$$h_1 - h_2 = -\int_1^2 vdp.$$

But $pv^\gamma = p_1 v_1^\gamma$, so that the last relation yields

$$h_1 - h_2 = \frac{\gamma}{\gamma - 1} p_1 v_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right].$$

By (6.15), and $p_1 v_1 = R^* T_1$

$$\frac{V_{s2}^2}{2g} = \frac{\gamma}{\gamma - 1} R^* T_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right].$$

We note that equations (6.12) and (6.13) are basically involved in the derivation of the equations above in $V_{s2}$ and $T_2$ respectively.

Now

$$v_2 = v_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}} \left(\frac{R^* T_1}{p_1}\right).$$

Since the weight rate $w$ of flow of the gas is such that

$$w = \frac{A V_{s2}}{v_2}$$

for the orifice area $A$, we have

$$w = \frac{A\sqrt{2g}}{\sqrt{R^* T_1}} \sqrt{\frac{\gamma}{\gamma - 1}} \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}} p_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{1}{2}}.$$

The maximum rate of flow $w$ can be obtained from the calculus in terms of the variable $p_2$. By the calculus the maximum $w$ is attained when

$$p_2 = p_1 \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}}. \tag{6.29}$$

The pressure $p_2$ in (6.29) is the *critical pressure* $p_c$ corresponding to $p_1$, and $p_c/p_1$ is the *critical pressure ratio.*

For air and diatomic gases for which $\gamma = 1.4$, under ordinary conditions, the critical pressure satisfies the relation

$$p_c = 0.53 p_1$$

approximately. For superheated steam the value 0.53 may be replaced by 0.546.

The speed $V_{s2}$ corresponding to the critical pressure is such that

$$V_{s2} = \sqrt{R^* g \gamma T_2},$$

which is precisely the speed of sound in an ideal gas. See Section 13.11.

If the pressure after the orifice is less than the critical pressure $p_c$, it is an experimental fact that $p_2 = p_c$, and $w$ has the maximum value. As the pressure after the orifice is decreased from the critical pressure the rate of flow $w$ does not decrease, but remains constant. That this follows from theoretical considerations will now be shown.

We suppose that the nozzle discharges into a region with pressure $p_3$ (at some distance from the nozzle). The pressure will then vary from $p_2$

to $p_3$ in a transition region as we leave the nozzle, or the pressure will be constant. Let us assume that the nozzle discharges into a conduit with the pressure $p_3$ at a cross section $A$ some distance from the nozzle, and let the pressure from the orifice to $A$ be $p_c$, so that $p_2 = p_3 = p_c$. There is then a corresponding maximum speed $V_{s2}$ of discharge at the orifice. Now decrease $p_3$. The pressure wave due to this decrease will travel at the speed of sound toward the orifice. Since the gas already travels with the speed of sound, the pressure wave will remain at the section $A$ as a standing wave, and the pressure from $A$ to the orifice will not decrease. The pressure $p_2$ will thus remain at $p_c$.

For nonperfect gases the concept of critical pressure still applies, but is no longer independent of the initial state.

### 6.12. Flow of gas into a vessel.

We shall treat the flow of gas into a vessel as in Figure 127. The gas flows from the supply at pressure $p_s$ through an orifice with outlet pressure $p_i$ into a vessel where the pressure is represented by $p_v$. Let $u_v(t)$ be the internal energy per unit weight of the gas in the vessel at the instant $t$, and $U(t)$ be the total internal energy of the gas in the vessel. It is assumed that conditions are uniform at the supply as well as in the vessel. We let $G_v(t)$ be the weight of the gas in the vessel. It is convenient to take this in moles. The internal energy in the vessel is then $U(t)$, where

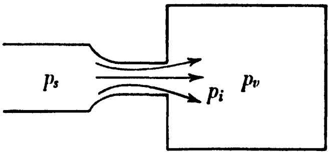


Fig. 127. Flow of a gas into a vessel.

$$U(t) = G_v(t)u_v(t).$$

This energy is changing at the rate $dU/dt$ given by

$$\frac{dU}{dt} = G_v\frac{du_v}{dt} + u_v\frac{dG_v}{dt}.$$

The derivatives here and to follow are assumed to exist. We neglect the heat exchange between the walls and the gas. The rate of change of internal energy of the gas in the vessel is equal to the rate at which internal energy is carried into the vessel by the entering gas, plus the rate at which work is done on the gas to push it into the vessel, plus the rate at which kinetic energy is carried into the vessel by the gas.

Let $u_i$ be the internal energy density per unit weight of the gas at the orifice outlet *i*. Internal energy is then carried into the vessel at the rate $dU_\iota/dt$, where

$$\frac{dU_\iota}{dt} = u_\iota \frac{dG_v}{dt}.$$

The rate of flow work necessary to push the gas into the vessel is

$$(p_\iota v_\iota - p_\iota v_\iota) \frac{dG_\iota}{dt}.$$

for the specific volume $v_\iota$ at the orifice outlet and the specific volume $v_v$ of the gas in the vessel. The gas actually diffuses in the vessel so that we do not strictly have flow of the incoming gas as a unit after the gas is in the vessel, and the flow work above is only an approximation.

Let $V_{s\iota}$ be the speed of the air entering the vessel, so that

$$\frac{V_{s\iota}^2}{2g}\left(\frac{dG_v}{dt}\right)$$

is the rate of kinetic energy flowing into the vessel. Equating the rate of internal energy increase to the rate of energy flow into the vessel we have

$$G_v \frac{du_v}{dt} = \left[(u_\iota - u_v) + p_\iota v_\iota - p_v v_v + \frac{V_{s\iota}^2}{2g}\right] \frac{dG_v}{dt}. \tag{6.30}$$

Let $h_\iota$ be the enthalpy of the gas at the vessel inlet, and $h_v$ the enthalpy of the gas in the vessel. We can write the above equation as

$$G_v \frac{du_v}{dt} = \left[(h_\iota - h_v) + \frac{V_{s\iota}^2}{2g}\right] \frac{dG_v}{dt}. \tag{6.31}$$

Let $V_0$ be the volume of the vessel. Then

$$V_0 = G_v v_v. \tag{6.32}$$

The mechanical and caloric equations

$$v = v(p, T), \qquad h = h(p, T) \tag{6.33}$$

of state are assumed known at each point in the system, where $v$, $p$, $T$, and $h$ are the specific volume, pressure, absolute temperature, and enthalpy, respectively. The enthalpy $h(p, T)$ is relative to some base state. We shall not assume that this is a function of $T$ alone because we do not wish to restrict our treatment to perfect gases.

We assume that the flow for the supply to the vessel inlet is adiabatic. As will be explained later in Section 7.9 the adiabatic $p,v$-relation for this flow is of the form

$$v = v(p) \tag{6.34}$$

for a function $v(p)$. By (6.34) we have

$$v_\iota = v(p_\iota). \tag{6.35}$$

We suppose that the supply gas is at rest. For flow to the vessel inlet we have

$$h_s - h_i = \frac{V_{si}^2}{2g}, \tag{6.36}$$

where $h_s$ is the enthalpy of the supply. Let $A_i$ be the orifice area. Then we have

$$\frac{dG_v}{dt} = A_i \frac{V_{si}}{v_i}, \tag{6.37}$$

so that

$$\frac{dG_v}{dt} = \frac{A_i \sqrt{2g}}{v_i} \sqrt{h_s - h_i}. \tag{6.38}$$

It is assumed that the state of the supply gas is known. We also have the equation

$$u_v = u_v(p_v, T_v) \tag{6.39}$$

for the absolute temperature $T_v$ of the gas in the vessel.

Let $T_i$ be the absolute temperature at the vessel inlet. We then have the equations

$$v_i = v(p_i, T_i), \qquad v_v = v(p_v, T_v) \tag{6.40}$$

for the specific volumes $v_i$ and $v_v$.

We also have the equations

$$h_i = h(p_i, T_i), \qquad h_v = h(p_v, T_v) \tag{6.41}$$

derived from (6.33). The equations given by (6.31), (6.32), (6.35), (6.36), (6.38)–(6.41) are *ten equations in the eleven variables*, $T_v$, $T_i$, $h_i$, $h_v$, $G_v$, $u_v$, $p_i$, $p_v$, $v_i$, $v_v$, and $V_{si}$. We need another relation to have a complete set of equations.

Let $p_c$ be the critical pressure relative to the pressure $p_s$, and let $r_c = p_c/p_s$ be the critical pressure ratio. If

$$\frac{p_v}{p_s} \geqq r_c,$$

we have

$$p_i = p_v. \tag{6.42}$$

When

$$\frac{p_v}{p_s} < r_c$$

we have

$$p_i = p_c. \tag{6.43}$$

The system is now mathematically determined. With (6.42) or (6.43), as the case may be, we have eleven equations in eleven unknowns.

To determine whether or not we have (6.42) or (6.43) at a given instant. we should compare $p_s$ with the solution $p_v$ of the eleven equations.

Since we have a dynamic system we need initial conditions. Thus we might suddenly open the orifice from a completely closed state with the vessel initially at a vacuum. Then at $t = 0$ we have $p_v = 0$, $p_i = p_c$. We first have ten equations in ten unknowns since $p_i$ is fixed until $p_v$ attains the critical value $p_c$. When $p_v$ reaches $p_c$ we adjoin the equation (6.42) to solve eleven equations for eleven unknowns.

If the filling of the vessel is a fairly rapid process, the assumption that we can neglect heat transfer between the gas and the walls is justified.

**6.13. Gas turbines.** We shall illustrate thermodynamic processes with an aircraft gas turbine as pictured in Figure 128. The turbine may have a centrifugal or axial-flow compressor (an axial flow compressor is shown in Figure 128). A gas turbine is generally designed by choosing in advance the power to be supplied by the turbine wheel. From experience and various considerations an angular speed is selected for the wheel, and a geometric design is tried on the basis of experience. The speed of the incoming air is known roughly in advance from the power required and the fact that a certain fuel-air ratio is desired. The compressor outlet pressure is then computed. This may be done as follows.

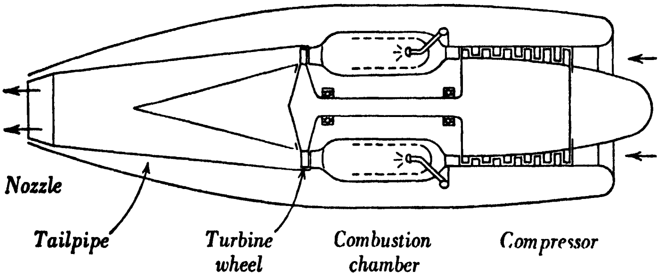


Fig. 128. Gas turbine.

We shall consider a single stage of the compressor. Let the values of the pressure $p$, specific volume $v$, absolute temperature $T$, enthalpy $h$, and speed $V$ of the air relative to fixed members of the gas turbine be labeled with the subscripts 1 and 2 to indicate values at the inlet and outlet of the compressor stage respectively. We suppose that the air flow between the blades is adiabatic or polytropic. The cross sectional area of a passage between the blades is varied along the passage and is intended to give a desired pressure drop across the compressor stage or air velocity at the outlet. Conditions at the stage inlet are taken as known. By the general gas law, and the adiabatic or polytropic nature of the flow between the blades, we have

$$T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}.$$

Since, up to a constant, the enthalpy $h$ is $c_pT$, equation (6.12) gives

$$W = Jc_p(T_2 - T_1) + \frac{V_2^2}{2g} - \frac{V_1^2}{2g},$$

where $W$ is the work in foot pounds done by the blades of the stage on the gas per pound of air, $J$ is Joule's constant, and $g$ is the magnitude of the acceleration of gravity. We often have

$$V_1 = V_2$$

approximately, so that

$$W = Jc_p(T_2 - T_1).$$

Angular momentum is the moment of inertia times the angular speed. The load torque per stage is the rate at which angular momentum leaves with the air minus the rate at which it enters, whence

$$W = \frac{uV_t - u_1V_{t1}}{g},$$

where $u$ is the peripheral speed of the blades, $V_t$ is the tangential component of $V_2$ (perpendicular to the radial component and the axis of the rotor), $V_{t1}$ is the tangential component of $V_1$, and $u_1$ is the linear speed of points on the compressor at the air inlet. For an axial flow compressor $V_{t1} = 0$, and for a centrifugal compressor $uV_t$ generally dominates $u_1V_{t1}$, so that the formula for W reduces to

$$W = \frac{uV_t}{g}.$$

The quantity $u$ is known from the rotational speed of the compressor. We can express $V_t$ in terms of $V_2$. From the polytropic or adiabatic law

$$v_2 = v_1\left(\frac{p_1}{p_2}\right)^{\frac{1}{n}},$$

the weight rate $K$ of air flow, and the cross sectional area $A$ of the compressor stage outlet passages we obtain $V_2$ as the function of $p_2$ given by

$$V_2 = \frac{Kv_1}{A_1}\left(\frac{p_1}{p_2}\right)^{\frac{1}{n}}.$$

Equating the expressions for $W$ we obtain an equation in $p_2$ which can be solved for $p_2$ by trial, whence the state of the gas at the outlet of the compressor stage is known.

A compressor generally has several stages. By repetition of the above process we can obtain the pressure at the outlet of the compressor. We can go on through the gas turbine with thermodynamic calculations and determine whether or not the correct power is delivered by the wheel, and whether or not the computation is consistent. If the turbine does not satisfy the required properties, it is necessary to run through the calculations again with some assumed properties changed.

Let us suppose that the compressor outlet pressure is known. We shall describe the calculations we can then perform to determine the states of the gas in the turbine. Unless great accuracy is required it is convenient to assume that the flow through the compressor is isentropic. We suppose that the air is a perfect gas. By the adiabatic law (6.25) or tables based on this law we can compute the enthalpy in British thermal units per pound at the compressor outlet measured from the enthalpy at some base state.

Fuel is mixed with the air at a certain ratio by weight, and the enthalpy of the fuel before mixing is known. The mixing can be supposed to take place adiabatically so that the enthalpy of the air-fuel mixture per pound is the sum of the enthalpy of the air in one pound of the mixture plus the enthalpy of the fuel in this pound. Since most of the mixture is air it is reasonable to treat it physically as if it were all air. The enthalpy of the mixture then corresponds to a different temperature from that of the air before mixing. We now add enthalpy due to combustion, known from experiment. The enthalpy of the burned mixture, now treated as air, will correspond to a certain combustion chamber temperature of the gas ready to pass through the wheel. We consider the expansion through the wheel to be isentropic. Knowing the temperature and pressure in the combustion chamber, and the pressure (around one atmosphere) in the tailpipe after the wheel, we can obtain the temperature and enthalpy after the wheel. If desired, we can compute the shaft work at the wheel from (6.17), from the given efficiency of the wheel, (which efficiency might be 70 per cent for example), from the difference between the enthalpy per pound of the mixture before and after the wheel, and from the pounds per hour of the mixture passing through the wheel (the latter found by computation or experiment). The actual tailpipe pressure after the wheel and the states of the gas as it flows through the tailpipe and outlet nozzle can be taken into account by assuming adiabatic flow.

The gas turbine cycle is pictured on a $p$, $v$-diagram in Figure 129 where the first stage is represented by the arc 12, and the last by the arc 41. Such diagrams are very useful. Here $v$ is specific volume, and $p$ the pressure of the gas. The line integral of $pdv$ around the curve gives the net work (output) for the cycle. This is simply the area of the region bounded by the curve.

Suppose now that we are concerned with the temperature deviation from an arbitrary steady state in the combustion chamber. We let $T$ and $N$ be, respectively, the combustion chamber temperature and revolutions per minute deviations from the steady state, and $W_f$ the deviation in mass fuel rate. It is readily seen that the state of the turbine depends on the variables $W_f$ and $N$ only, assuming constant ambient conditions, which is true in practice for reasonably long intervals of time. The combustion chamber temperature is a function of $W_f$ and $N$ which obviously increases with $W_f$ and decreases with $N$. Unless the deviations from the reference state are too large we can assume that

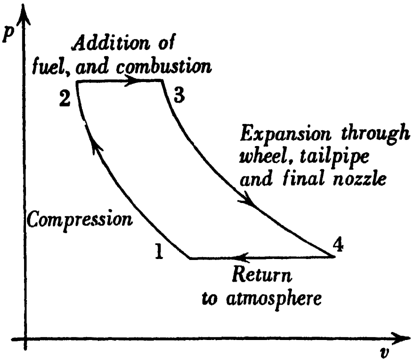

p
Addition of fuel, and combustion
2
3
Expansion through wheel, tailpipe and final nozzle
Compression
1
4
Return to atmosphere
v

Fig. 129. Gas turbine cycle.

$$T = aW_f - bN$$

for positive numbers $a$ and $b$. The net torque which drives the wheel-compressor combination will increase with the fuel rate and decrease with the speed so that we have

$$IN' = cW_f - eN$$

for the moment of inertia $I$ of the rotating mechanism with respect to the shaft axis, the acceleration $N'$ of the wheel, and positive constants $c$, $e$. The constants $a$ and $-b$ are $\partial T/\partial W_f$ and $\partial T/\partial N$. The constants $c$ and $-e$ are partial derivatives of torque with respect to $W_f$ and $N$ respectively.

The partial derivatives $a$, $b$, $c$, and $e$ can be evaluated from the operating characteristics of the wheel obtained by calculation or experiment.

Efficiencies of 85 per cent for the gas turbine compressor and the turbine wheel are not unreasonable. For accurate results we must assume a combustion efficiency, such as 90 per cent, and a turbine nozzle efficiency, for example 90 per cent.

Due to the high pressure in the fuel line, such as 1,000 pounds per square inch, the rate of fuel flow is practically independent of the gas turbine combustion chamber pressure. This chamber pressure might, for example, be five or six atmospheres.

### 6.14. Rocket.

In Figure 130 we have pictured a rocket in rough schematic form. In this rocket alcohol is burned in oxygen, both alcohol and oxygen being supplied from a liquid reserve in the rocket. The resulting gases pass through a jet into the atmosphere. In Section 2.10 we gave an approximate treatment of the mechanical properties of a rocket. These properties depend on thermodynamic considerations to be given in this section.

We let $W_a$ and $W_o$ be the mass rate of flow of the alcohol and the oxygen into the combustion chamber of the rocket. The alcohol and oxygen being under high pressure, $W_a$ and $W_o$ are for practical purposes independent of conditions in the combustion chamber, and will be assumed to be given.

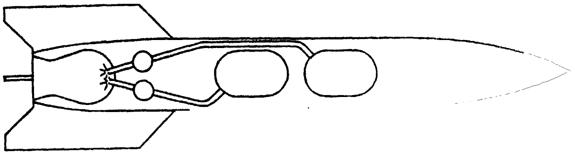


Fig. 130. Rocket.

The proportion of $W_a$ to $W_o$ is fixed. We therefore consider $dm/dt$ where $m$ is the mass of the alcohol and oxygen fuel already discharged from the rocket at the instant under consideration.

The gas masses travel out of the rocket with tremendous speed, so that storage effects in the combustion chamber can be neglected. We can therefore suppose that the mass rate of fuel fed to the combustion chamber is equal to mass rate of ejection of particles from the rocket, that is $dm/dt$. The quantity $dm/dt$ is then understood to be given.

We let the subscript 1 designate the state within the combustion chamber ahead of the exit nozzle, and 2 the state after the exit nozzle. To simplify the discussion we shall analyze the rocket while it is traveling directly upward from the earth. If the rocket is not in this motion, we need merely introduce components. We let $x$ be the vertical coordinate of a point on the rocket geometrically defined in terms of the rocket, the positive direction for $x$ being upward. We shall suppose that the attitude of the rocket with respect to the vertical is constant.

We let $p$, $V_s$, $v$, and $T$ denote pressure, speed relative to the rocket, specific volume, and absolute temperature of the gas, respectively. The instantaneous mass $M$ of the rocket is the difference between the initial mass $M_o$ of the rocket and the mass $m$ already discharged. Let $F$ denote the external force on the rocket. We shall understand that the term "rocket" refers to the rocket and contents up to the end surface of the nozzle.

The *rate of change of momentum of the rocket is the rate at which momentum is lost through discharge of gases in the jet plus the rate of change of momentum of the rocket mass,* as can be seen as follows for the case under consideration. At an instant $t$ we take the momentum $N$ of the rocket mass to be $(M_0 - m)x'$, where the prime denotes differentiation with respect to time. At the instant $t + \Delta t$ the mass of the rocket is $M_0 - m - \Delta m$ for a quantity $\Delta m$, and the momentum of this mass is

$(M_0 - m - \Delta m)(x' + \Delta x')$ for a quantity $\Delta x'$. The momentum of the particles ejected during the time $\Delta t$ is $(x' - V_{s2})\Delta m$, the velocity of the ejected particles being $x' - V_{s2}$ relative to the earth. The total momentum $N + \Delta N$ at the instant $t + \Delta t$ is $(M_0 - m - \Delta m)(x' + \Delta x') + (x' - V_{s2})\Delta m$. The change $\Delta N$ in the momentum during the time $\Delta t$ is thus $(M_0 - m)\Delta x' - \Delta m \Delta x' - V_{s2}\Delta m$. Dividing by $\Delta t$ and taking the limit as $\Delta t \to 0$, we obtain

$$\frac{dN}{dt} = (M_0 - m)x'' - V_{s2}m'.$$

We can write this equation as

$$\frac{dN}{dt} = (x' - V_{s2})m' + \frac{d(Mx')}{dt}.$$

The first term on the right in this equation is the rate at which momentum is lost through the nozzle. By Newton's second law of motion we have

$$(x' - V_{s2})\frac{dm}{dt} + \frac{d(Mx')}{dt} = F. \tag{6.44}$$

Now

$$F = -(M_0 - m)g - F_f$$

for the function $F_f$ representing the drag force on the rocket. Relation (6.44) can be written as

$$M_0x'' - mx'' - V_{s2}m' = (m - M_0)g - F_f. \tag{6.45}$$

The drag is due to two factors, the first being the drag $F_a$ of the air on the rocket and the second being the drag $F_g$ from the friction of the gases on the internal wall of the rocket. From Part V we know that $F_a$ numerically is an increasing function of $x'$, and for practical purposes can be approximated by $cx'$ for a constant $c$. The drag $F_g$ can similarly be taken to be a function of the speed $V_{s2}$ of the exhausting gases relative to the rocket.

It will be seen that the remaining equations derive from a study of the gases traveling through the rocket.

The enthalpies of the liquid alcohol and oxygen referred to some base state are assumed known from experiment, and the temperature and pressure of these liquids entering the combustion chamber are supposed to be known from calculations readily performed with the aid of the laws in this book. Evaporating at the chamber pressure $p_1$ and at the temperature of the liquid, the increase in enthalpy can be computed from the latent heats of the alcohol and oxygen as a function of $p_1$. The resulting gases are mixed, the enthalpy of the mixture being directly obtainable from the enthalpies of the component vapors and the proportion of these components. Thus if one pound of the mixture contains $x$ pounds of the alcohol and $y$ pounds of the oxygen ($x$ and $y$ are fractions less than 1), and $H_a$

is the enthalpy of the alcohol vapor, $H_0$ the enthalpy of the oxygen, then the enthalpy of the mixture is $(xH_a + yH_0)$. That is, *the enthalpies of the components of a mixture* here contribute only to the enthalpy of the mixture, and do not change in whole or part to another form of energy. To the enthalpy of the mixture must be added the heat of combustion per pound of the mixture. The final enthalpy corresponds to a unique temperature $T_1$ in the combustion chamber, so that we have

$$T_1 = f(p_1) \tag{6.46}$$

for a known function $f(p_1)$ of $p_1$.

The general gas law gives

$$p_1v_1 = RT_1, \tag{6.47}$$

for the specific volume $v_1$ of the burned gas in the combustion chamber, and a constant $R$.

The gas expands through the nozzle of the tail adiabatically so that

$$p_1v_1^\gamma = p_2v_2^\gamma \tag{6.48}$$

for the adiabatic exponent $\gamma$, the pressure $p_2$, and specific volume $v_2$ at the exit.

For great accuracy it is necessary to allow for the efficiency of combustion, probably on the order of 90 per cent. Since the gases issuing from the rocket have some of the original fuel vapors, the enthalpies must be corrected accordingly. Similarly a jet efficiency, such as 90 per cent should be used for a polytropic process instead of that for (6.48).

Let $V_{s1}$ be the speed of the gases at the entrance to the nozzle relative to the nozzle. Just where the nozzle entrance is located must, of course, be defined. We assume that this has been done. We denote the effective orifice area at the entrance to the nozzle by $A_1$, and the corresponding area at the exit by $A_2$. By *effective* is meant the area multiplied by a corrective factor to take care of efficiency, where the efficiency might, for example, be 90 per cent. The mass rate of flow of the gas past the nozzle exit is then $A_2(V_{s2}/v_2)$ so that

$$\frac{dm}{dt} = \frac{A_2V_{s2}}{v_2}. \tag{6.49}$$

Assuming no storage in the nozzle (since the mass rate of storage is certainly small compared to the mass rate of flow of the gas relative to the rocket) we have the equation

$$\frac{A_1V_{s1}}{v_1} = \frac{A_2V_{s2}}{v_2} \tag{6.50}$$

We shall apply Newton's second law of motion to the gas in the nozzle, which we may assume for practical purposes to be in a steady state. We suppose for simplicity, and contrary to practice, that the nozzle simply narrows continuously down from section 1 to section 2. Except for friction, which we neglect for the moment, the net force on the gas in the nozzle in the vertical direction is approximately $(-p_1A_1 + p_2A_2^*)$ for an *equivalent* area $A_2^*$ in place of $A_2$ due to the pressure on the narrowing walls of the nozzle. The density of the momentum at each point in the nozzle is assumed constant during a time $\Delta t$. The momentum of the gas leaving the nozzle is given by $(x' - V_{s2})\Delta m$ for the mass $\Delta m$ leaving the rocket in the time $\Delta t$. The momentum of the gas entering the nozzle is $(x' - V_{s1})\Delta m$. The net change in momentum is $(V_{s1} - V_{s2})\Delta m$, and the instantaneous rate of change of momentum is $(V_{s1} - V_{s2})dm/dt$, so that Newton's law gives

$$(p_2A_2^* - p_1A_1) = (V_{s2} - V_{s1})\frac{dm}{dt}. \tag{6.51}$$

An extra term can be included in (6.51) to take care of friction, such as one proportional to the average of $V_{s1}$ and $V_{s2}$.

For greater accuracy we can develop an equation like (6.51), but valid at each cross section of the rocket perpendicular to its axis. Since it is convenient to make a rocket symmetric, such an axis exists.

There are seven equations (6.45) to (6.51) in the seven variables $p_1$, $V_{s1}$, $v_1$, $T_1$, $v_2$, $V_{s2}$, and $x$, so that the desired system of equations is secured. The atmospheric pressure may be less than critical relative to $p_1$, whence the corresponding critical pressure $p_c$ should be used in place of $p_2$. If the atmospheric pressure is not less than critical, the pressure $p_2$ is atmospheric and is known. If we are to allow for a variable rate $dm/dt$, we can suppose that $dm/dt$ is a given function of $t$, inasmuch as $dm/dt$ will probably change according to a known law. Since this derivative will undoubtedly change relatively slowly the equation (6.51) is still valid.

If $m$ is small compared to $M_0$, as is true near the start of flight, we can drop the second term in (6.45) to get

$$M_0x'' = V_{s2}m' + F. \tag{6.52}$$

Equation (6.52) is the equation of motion of a body with mass $M_0$ subject to the thrust $V_{s2}m'$. Actually, by (6.44) the thrust on the rocket is

$$(V_{s2} - x')m'.$$

If, however, $V_{s2}$ is large compared to $x'$, we can take this thrust to be $V_{s2}m'$. Consider the $V$-2. Unless the rocket has attained a tremendous speed, we will have $V_{s2}$ dominating $x'$. See Figure 131 for a $V$-2.

### 6.15. Gasoline and diesel engines.

The process in a four-cycle gasoline engine can be approximated by the Otto cycle of Figure 132 when $p$ is the pressure in the cylinder and $V$ is the cylinder volume. We suppose for rough calculations, that the gas in the cylinder can be treated

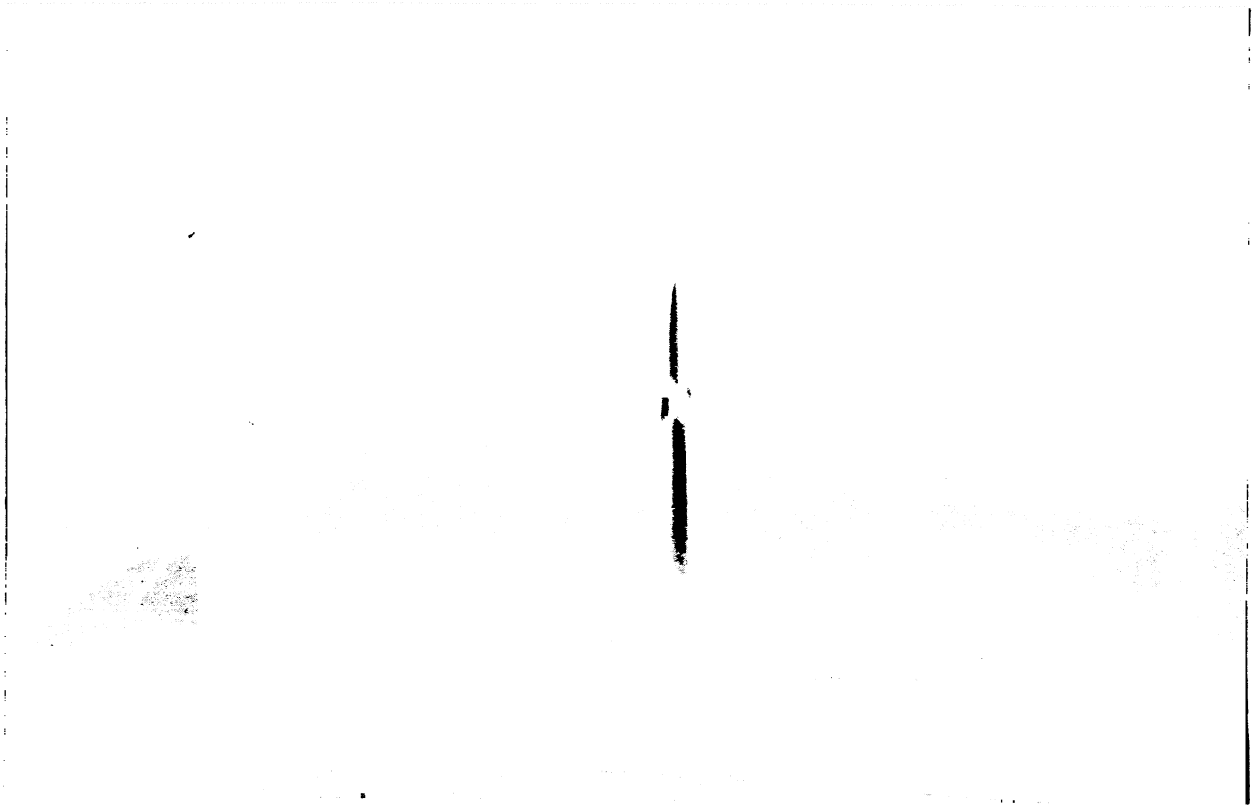


Fig. 131. A V-2 rocket taking off at White Sands, New Mexico. Courtesy of the United States Army Air Force.

as air at least where the gas laws are concerned. The line segment 12 represents the isobaric intake of the suction stroke. The pressure is practically atmospheric. The mass of gas in the cylinder increases so that the *line segment 12 does not represent the varying states of a given mass of gas*, but rather the instantaneous state of the gas in the cylinder, where the mass of the gas is varying. The curve 23 corresponds to the adiabatic compression of the gas in the cylinder. We are neglecting here the nonuniformity of the gas, friction, and conduction of heat through the walls. The line segment 34 represents the explosion, which takes place at constant volume.

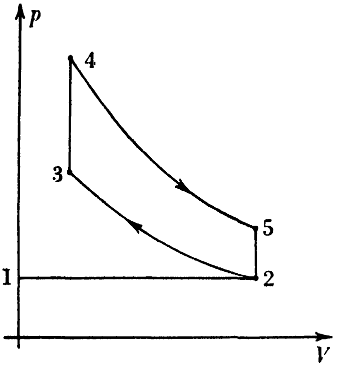


Fig. 132. Otto cycle.

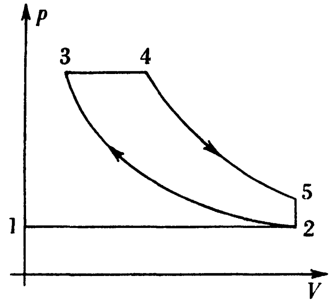


Fig. 133. Diesel cycle.

To compute the enthalpy of the state 4 we need but add the enthalpy of combustion to the enthalpy at state 3. The curve 45 corresponds to the adiabatic expansion of the gas in the cylinder as the piston makes its power stroke. The line segment 52 indicates the drop in pressure when the exhaust valve opens. This drop is actually accomplished by the loss of some of the cylinder gas to the atmosphere. The isobaric exhaust stroke is taken care of by going from 2 back to 1 along the line segment 12, whence the cycle is completed. In computations on the gasoline engine it is often unnecessary to treat the states from 1 to 2.

In Figure 133 we have indicated a four-cycle diesel process. This differs from the Otto cycle of Figure 132 primarily in that combustion takes place at practically constant pressure, while the piston is moving in the cylinder expansion direction during part of the power stroke.

The compression ratio of the engine is known from the geometry of the engine.

### 6.16. One cylinder gasoline engine during intake.

In the previous section it is assumed that the pressure in the cylinder is atmospheric during intake. Actually, the cylinder pressure is less than atmospheric. We shall now take this factor into account, and develop the equations which describe the motion.

The rotating parts will be referred to an angle $\theta$ which the crank makes with the vertical (see Figure 134). The same notation will be used for the state of the gas mixture in the cylinder as in Section 6.12 where we treated the flow of a gas into a vessel; that is, $p_v$, $T_v$, $v_v$, $u_v$, and $G_v$ will be used for

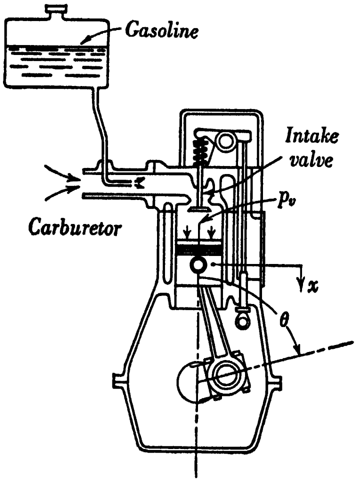


**Fig. 134. One cylinder gasoline engine during intake.**

the pressure, absolute temperature, specific volume, internal energy density, and weight of the gas in the cylinder, respectively. We shall use the same notations for the intake valve outlet and supply variables, namely, $p_i$, $v_i$, $u_i$, $T_i$, $h_i$, and $V_{si}$ for the valve outlet pressure, specific volume, internal energy density, absolute temperature, enthalpy, and speed, and the subscript $s$ in place of $i$ for corresponding supply variables.

We let $w_a$ and $w_g$ be the respective rates of flow by weight of the air and the gasoline into the cylinder, so that

$$\frac{dG_v}{dt} = w_a + w_g.$$

We shall suppose that most of the pressure drop is across the cylinder valve and shall therefore neglect the drop between the atmosphere and the carburetor. The carburetor pressure is then atmospheric pressure $p_s$. We suppose that the mixing of the gasoline and the air takes place at the carburetor. To simplify the argument we shall assume that it is known that $w_g$ is a definite proportion of $w_a$. Let $h_a$ and $h_g$ be the enthalpies of the air and gasoline at atmospheric temperature and pressure. Assuming that the mixing of the gasoline and air takes place at constant pressure and temperature, the enthalpy $h_s$ of the mixture is given by

$$(w_a + w_g)h_s = w_a h_a + w_g h_g.$$

If $u_a$, $u_g$, and $u_s$ are, respectively, the internal energy densities of the air, the gasoline, and the mixture of the air and the gasoline at atmospheric pressure and temperature, there is the corresponding relation

$$(w_a + w_g)u_s = w_a u_a + w_g u_g,$$

which will not be explicitly needed for what follows.

The supply pressure is the pressure $p_s$ of the atmosphere. We neglect the throttle for which there is but little enthalpy change. We let $A$ be the area of the cylinder piston. For an easily determined function $x(\theta)$ the coordinate of position of the piston is related to the angle $\theta$ by the relation

$$x = x(\theta), \tag{6.53}$$

and the volume $V_0$ of the cylinder satisfies a relation

$$V_0 = V(\theta) \tag{6.54}$$

for a known function $V(\theta)$.

The equations of Section 6.12 now apply except that the energy equation (6.30) must be corrected for the movement of the piston. The power associated with the movement of this piston is

$$Ap_v \frac{dx}{dt},$$

where $x$ increases in the downward direction, so that equation (6.30) becomes

$$G_v \frac{du_v}{dt} = \left[(h_i - h_v) + \frac{V_{si}^2}{2g}\right] \frac{dG_v}{dt} - Ap_v \frac{dx}{dt}. \tag{6.55}$$

We let $I(\theta)$ be the moment of inertia of the rotating parts and members rigidly connected to the crankshaft such as the piston, with respect to the axis of rotation. Further, we let $\tau(\theta, p_v)$ be the torque on the crankshaft due to cylinder pressure, $L$ the load torque, and $F(\theta')$ the friction and windage torque, the prime denoting derivative with respect to time. We then have

$$I(\theta)\theta'' = \tau(\theta, p_v) - F(\theta') - L \tag{6.56}$$

as the torque equation.

The driving torque $\tau(\theta, p_v)$ can be derived from the force $(p_v - p_s)A$ acting on the piston, the geometry involved, and the fact that if we neglect friction, the force $(p_v - p_s)A$ is balanced by the reaction of the connecting rod on the piston.

We now have 14 equations in (6.32), (6.35), (6.36), (6.38)–(6.41), (6.53)–(6.56), and (6.42) or (6.43) as the case may be. These are 14 equations in the 14 unknowns $u_v$, $v_v$, $v_i$, $T_v$, $T_i$, $p_v$, $p_i$, $h_v$, $h_i$, $G_v$, $V_{si}$, $V_0$, $\theta$, and $x$.

The remaining strokes of the engine are easier to analyze.

If we wish to take account of such factors as heat conduction through the cylinder wall, the system of equations obtained above must be modified accordingly. It is not difficult to extend the results of this section to an actual engine, and to an engine with more than one cylinder.

**6.17. Steam.** The techniques developed in this chapter apply to steam as well as other vapors. We must often be concerned with the *quality* of the steam, where quality is defined as follows: in a mixture of a liquid and vapor the quality is the per cent by weight that is vapor. Extensive tables of the thermodynamic properties of steam are available.

If the temperature of a vapor is much higher than condensation temperature, or if the pressure of the vapor is very low, it may be treated as a perfect gas; otherwise this is not possible. For steam, various equations of state relating pressure, volume, and temperature have been obtained experimentally, and corresponding sets of values may be found in tables. That part of the thermodynamic theory developed above which is not based on the general gas law applies to steam.

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# CHAPTER

# 7

# *Second law of thermodynamics*

**7.1. The Kelvin-Planck and Clausius formulations of the second law of thermodynamics.** From the first law of thermodynamics we gather that heat can be converted into an equivalent amount of work. From the second law we know that we cannot do this without making other changes in the given system and the environment. To formulate the second law of thermodynamics it is convenient to introduce some terms.

An *engine* is a device that does work. We shall consider engines that operate with working substances, such as air or steam, where the working substance goes from a state $A$ through intermediate states back to the state $A$, this process being repeated over and over. Such an engine is said to operate in a *cycle*. A steam turbine combined with a boiler and condenser operates in a cycle. Here water is constantly recirculated.

A *reservoir* is a body to which any amount of heat can be added, or from which any amount of heat can be extracted, without changing the temperature of the body. The ocean is an example of a body which can, for practical purposes, be generally considered a reservoir. The boiler supplying a steam turbine is another example of a reservoir. Whether or not a physical body can be treated as a reservoir depends on the processes studied.

*Law* **7.1. (Kelvin-Planck form of the second law of thermodynamics).** *It is impossible to construct an engine that, operating on a cycle, will extract heat from a reservoir, and produce an equivalent amount of work (such as raising a weight) without other thermodynamic effects.*

*Law* **7.2. (Clausius' form of the second law of thermodynamics).** *It is impossible to construct a device that, operating on a cycle, will transfer heat from a cooler to a hotter reservoir with no other effects on the system and its environment.*

A device that operates on a cycle, extracts heat from a cold reservoir, and adds heat to a hot reservoir, is called a *refrigerator*. By use of engines and refrigerators we shall show that Laws 7.1 and 7.2 are equivalent. This can be accomplished by showing that when one of the laws is violated, the same is true of the other.

Suppose that Law 7.2 is violated, and that we have a refrigerator that during a cycle transfers heat $Q_2$ from a cold reservoir to a hot reservoir, without work being done on the refrigerator. Assume that an engine $E$ operates between the same two reservoirs and during a cycle delivers heat $Q_2$ to the cold reservoir, while extracting heat $Q_1$ from the hot reservoir, and doing an amount of work $W$ ($\neq 0$), where $W = Q_1 - Q_2$. Suppose that the engine and refrigerator go through a cycle simultaneously. The refrigerator and engine together extract heat $(Q_1 - Q_2)$ from the hot reservoir, and convert this into work. We may suppose that this occurs without other thermodynamic changes. Law 7.1 is now contradicted.

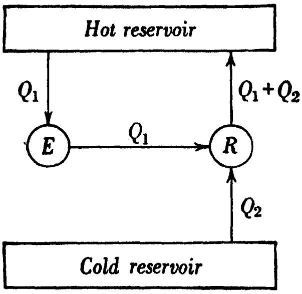

**Fig. 135. Impossible engine-refrigerator combination.**

Suppose that Law 7.1 is violated, and that an engine during a cycle draws heat $Q_1$ from a reservoir (called *hot*) and turns this directly into work $W$ ($\neq 0$). We use this work to drive a refrigerator $R$, which during a cycle draws some amount of heat $Q_2$ from a cold reservoir, and delivers the heat $(Q_1 + Q_2)$ to the hot reservoir, using up the work $W$ (see Figure 135). Assume that the engine and refrigerator go through a cycle simultaneously. The engine-refrigerator combination transfers heat $Q_2$ from the cold reservoir to the hot reservoir without other changes in the system and its environment; this violates Law 7.2.

The second law of thermodynamics implies, for example, that it is impossible to operate an engine on a cycle by merely extracting heat from the earth, and turning this heat directly into mechanical work or some other form of work.

We shall derive mathematical consequences of the second law of thermodynamics in sections to follow.

The laws of thermodynamics are important, in particular, in that we can determine basic limitations on the performance of equipment of various designs, and determine the optimum performance of a given design.

A mathematical advantage of an engine operating on a cycle is that a thermodynamic relation for a cycle does not involve internal energy. Thus if we start a cycle with a state $A$, we return to $A$ upon the completion of the cycle so that $U_B = U_A$ in (6.7), and (6.7) becomes simply $Q = W$.

In the above proof we have assumed the existence of the engine $E$ and refrigerator $R$ regardless of the magnitudes of $Q_2$ and $W$ respectively.

### 7.2. Carnot cycle.

In a power cycle some heat is absorbed from a hot reservoir during one part of the cycle, and a smaller amount is given to a cold reservoir during another part, the work done being equivalent to the difference between these heats. Thus in a steam power cycle some heat comes from the boiler, which is the hot reservoir, and some of this heat is discharged to the condenser, which is the cold reservoir. We shall show that the most efficient engine operating on a cycle between hot and cold reservoirs is one that operates on a reversible cycle of which the Carnot cycle is an example.

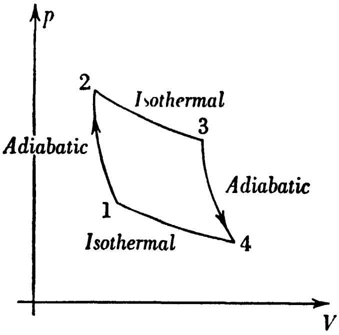

Fig. 136. Carnot cycle.

A *Carnot cycle* is a process with a working substance first in thermal equilibrium with a cold reservoir at a temperature $T_1$. A reversible adiabatic process is performed on the substance so that the temperature of the substance rises to the temperature $T_2$ of a hot reservoir. The working substance is kept in contact with the hot reservoir during a reversible isothermal process while a quantity of heat $Q_2$ is absorbed from the reservoir. A reversible adiabatic process is now accomplished until the temperature of the working substance drops to $T_1$. The substance is kept in contact with the cold reservoir during a reversible isothermal process until the substance returns to its initial state. The cycle is now completed. During contact with the cold reservoir the substance delivers some heat $Q_1$ to this reservoir.

In Figure 136 we have pictured a Carnot cycle for a gas. The gas is compressed adiabatically from 1 to 2 so that the gas goes from the temperature $T_1$ to the temperature $T_2$. The gas then expands isothermally at the temperature $T_2$ from 2 to 3, absorbing heat from a hot reservoir. The gas then expands adiabatically until the temperature $T_1$ is reached at 4. The gas is now compressed at the temperature $T_1$, rejecting heat to a cold reservoir until the point 1 is reached again, and the cycle repeats.

If we perform the Carnot cycle in reverse order we have a *Carnot refrigerator.*

**7.3. Engine operating between two reservoirs.** An engine operating on a cycle between two reservoirs must extract heat from the hotter reservoir, and deliver heat to a colder reservoir, as the following argument shows.

Suppose, on the contrary, that an engine $E$ during a cycle extracts heat $Q_2$ from a cold reservoir $C$ and transfers heat $Q_1$ to a hot reservoir $H$, as in Figure 137. To be an engine, $E$ must deliver a positive amount of work $(Q_2 - Q_1)$, so that $Q_2 > Q_1$. We use the engine to drive a perfectly feasible refrigerator $R$ which during a cycle draws some heat $Q_3$ from $C$, and thus delivers heat $(Q_2 - Q_1 + Q_3)$ to $H$. Assume that the engine and refrigerator combination go through a cycle simultaneously. The engine-refrigerator combination does nothing except transfer heat $(Q_2 + Q_3)$ from $C$ to $H$, which contradicts the second law of thermodynamics.

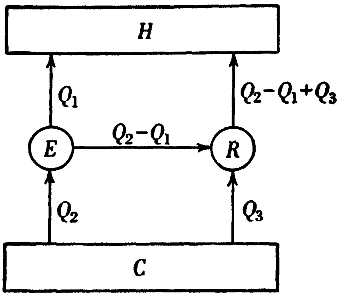

**Fig. 137. An impossible configuration.**

Since an engine puts out work it follows that an engine operating between two reservoirs must extract a larger amount of heat from the hot reservoir than it discharges to the cold reservoir.

**7.4. Carnot cycle as a most efficient cycle.** To prove that a reversible cycle is a most efficient cycle operating between two reservoirs, let us compare an engine $A$ operating on a reversible cycle with a second engine $B$. Let us suppose that these engines both produce the same work $W$. We shall assume that during a cycle the engine $A$ absorbs heat $Q_A$ from the hot reservoir, and rejects $(Q_A - W)$ to the cold reservoir. By definition the *efficiency* of $A$ is $\eta_A$ given by $W/Q_A$.

Let us suppose that during a cycle the engine $B$ absorbs heat $Q_B$ from the hot reservoir, whence it rejects $(Q_B - W)$ to the cold reservoir. The engine $B$ has the efficiency $\eta_B$ equal to $W/Q_B$.

Suppose that

$$\eta_B > \eta_A.$$

It follows from the formulas for the efficiencies that

$$Q_A > Q_B. \tag{7.1}$$

We now use the work of the engine $B$ to drive the engine $A$ backward as a refrigerator $R$. We can drive $A$ backward because the cycle for $A$ is reversible. During a cycle the refrigerator $R$ now delivers heat $Q_A$ to the hot reservoir, and extracts heat $(Q_A - W)$ from the cold reservoir. During a cycle of the engine-refrigerator combination the net heat extracted from the cold reservoir is $[(Q_A - W) - (Q_B - W)]$, which is $(Q_A - Q_B)$. By (7.1) this difference is positive.

The total heat delivered to the hot reservoir during a cycle is the same difference $(Q_A - Q_B)$. Thus the engine $B$-refrigerator $R$ combination does nothing except transfer heat from a cold reservoir to a hot reservoir. This contradicts the second law of thermodynamics. Thus

$$\eta_B \leqq \eta_A,$$

as was to be proved.

Since a Carnot engine operates on a reversible cycle, the Carnot cycle is an example of a most efficient cycle.

It follows that any two engines operating on reversible cycles between the same two reservoirs have the same efficiency.

Consider a path $C$ joining points $L$ and $M$ in the $pv$-plane of a substance, and a physical process corresponding to going from $L$ to $M$ along $C$. By (6.7)

$$Q_{LM} = U_M - U_L + W_{LM},$$

where $Q_{LM}$ is the heat transferred during the process, $W_{LM}$ is the work done during this process, $U_L$ is the internal energy of the substance at the point $L$, and $U_M$ is the internal energy of this substance at the point $M$. Suppose that we reverse the process by going from $M$ to $L$ along the path $C$. By (6.7)

$$Q_{ML} = U_L - U_M + W_{ML},$$

where $Q_{ML}$ and $W_{ML}$ are the heat and work associated with the process. Since the work is the integral of $pdv$ along the path, it follows from the calculus that $W_{ML}$ is the negative of $W_{LM}$. As a consequence $Q_{ML}$ is the negative of $Q_{LM}$. Thus *the heat and work associated with going from the state $L$ to the state $M$ along $C$ change sign when we go from $M$ to $L$ along $C$.* Although we did not need this result for the proof above, we shall use it later.

**7.5. Kelvin temperature.** The Carnot cycle enables us to give a fundamental mathematical treatment of temperature.

We can measure the pressure $p$ and specific volume $v$ of a substance, assumed to be in a state of *thermal equilibrium.* We might use other thermodynamic variables than $p$ and $v$. If two bodies are placed in contact with each other, and are thermally insulated from the environment, we know from experience that these bodies will eventually reach thermal equilibrium.

For the definition of Kelvin temperature we shall assume that we have reservoirs available at all temperatures. We can tell whether or not two such reservoirs $R_1$ and $R_2$ are at the same temperature by placing a nonreservoir body $A$ in contact with $R_1$, where $A$ and $R_1$ are thermally insulated from the environment. After $A$ reaches thermal equilibrium place $A$ in contact with $R_2$ immediately and have $A$ and $R_2$ thermally insulated from the environment. If $A$ remains in the same thermal equilibrium, the reservoirs $R_1$ and $R_2$ are at the same temperature; otherwise not. We can assign a value of a variable $\theta$ (we assume that one variable is enough) to bodies at the same temperature, so that for bodies with different temperatures the corresponding values of $\theta$ are different. Then $\theta$ denotes temperature.

In Section 6.5 we introduced the concept of internal energy by thermally insulating a body from the environment and doing work on it. We can arrive at the concept of heat in terms of internal energy and work through equation (6.7) in the statement of the first law of thermodynamics. We therefore assume in what follows that we can measure the quantities of heat involved.

Let $R_1$ and $R_2$ be two reservoirs at temperatures $\theta_1$ and $\theta_2$, where $\theta_1 \neq \theta_2$. Suppose that a Carnot engine during one cycle absorbs $Q_1$ units of heat from $R_1$ and delivers $Q_2$ units of heat to $R_2$. The efficiency $\eta_{12}$ of the cycle is

$$\eta_{12} = \frac{Q_1 - Q_2}{Q_1}.$$

We can write

$$\eta_{12} = 1 - \frac{Q_2}{Q_1}.$$

The quantity $\eta_{12}$ is independent of the nature of the working substance; in fact $\eta_{12}$ depends only on the temperatures $\theta_1$ and $\theta_2$, from which it follows that

$$\frac{Q_1}{Q_2} = g(\theta_1, \theta_2) \tag{7.2}$$

for a function $g(\theta_1, \theta_2)$. Let $\theta_3$ be a value of $\theta$ distinct from $\theta_1$ and $\theta_2$, and suppose that a Carnot engine, operating between $R_2$ and a reservoir $R_3$ at the temperature $\theta_3$, during one cycle absorbs heat $Q_2$ from $R_2$, and rejects a certain amount of heat $Q_3$ to $R_3$. In this case

$$\frac{Q_2}{Q_3} = g(\theta_2, \theta_3). \tag{7.3}$$

Since the heat $Q_2$ rejected by the first Carnot engine is absorbed by the second Carnot engine, these two engines together can be combined to form a third Carnot engine that during a cycle absorbs heat $Q_1$ from $R_1$, and discharges heat $Q_3$ to $R_3$. Thus

$$\frac{Q_1}{Q_3} = g(\theta_1, \theta_3). \tag{7.4}$$

From (7.2)–(7.4) there follows

$$g(\theta_1, \theta_2) = \frac{g(\theta_1, \theta_3)}{g(\theta_2, \theta_3)}. \tag{7.5}$$

Since $R_3$ is rather arbitrary, we consider (7.5) to be an identity in $\theta_3$. To simplify the notations in the argument to follow, we write

$$\theta_1 = x, \qquad \theta_2 = y, \qquad \theta_3 = z.$$

In this notation (7.5) becomes

$$g(x, y) = \frac{g(x, z)}{g(y, z)}. \tag{7.6}$$

We suppose that the $\theta$-temperature scale can be chosen so that $\partial g(x, z)/\partial z$ exists and is a continuous function of $z$. Differentiating both sides of (7.6), the equality

$$\frac{\dfrac{\partial g(x, z)}{\partial z}}{g(x, z)} = \frac{\dfrac{\partial g(y, z)}{\partial z}}{g(y, z)}$$

is obtained. Thus

$$\frac{\dfrac{\partial g(x, z)}{\partial z}}{g(x, z)} = h(z) \tag{7.7}$$

for a function $h(z)$. Since the left side of (7.7) is the derivative of $\ln g(x, z)$ with respect to $z$,

$$\ln g(x, z) = \lambda(z) + \ln f(x)$$

for functions $f(x)$ and $\lambda(z)$, where

$$\lambda'(z) = h(z).$$

We can now write

$$g(x, z) = f(x)e^{\lambda(z)},$$

so that by (7.6)

$$g(x, y) = \frac{f(x)}{f(y)},$$

from which

$$\frac{Q_1}{Q_2} = \frac{f(\theta_1)}{f(\theta_2)}.$$

By definition the Kelvin temperatures $T_1$ and $T_2$ of $R_1$ and $R_2$ are such that

$$\frac{T_1}{T_2} = \frac{f(\theta_1)}{f(\theta_2)},$$

from which

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}. \tag{7.8}$$

Thus *two Kelvin temperatures are to each other as the heats absorbed and rejected by a Carnot engine operating between reservoirs at these temperatures.*

Since the $Q$'s are positive quantities in the above argument, the $T$'s must have the same sign. We therefore take the $T$'s non-negative. We note that since $Q_1 > Q_2$ for the reservoirs $R_1$ and $R_2$ treated above, we also have $T_1 > T_2$.

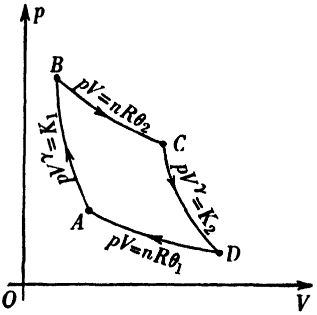

**Fig. 138. Carnot cycle of a perfect gas.**

We let $T_s$ and $T_i$ be the Kelvin temperatures at the steam and ice points respectively. By definition

$$T_s - T_i = 100.$$

If a Carnot engine rejects no heat to the reservoir at the lower temperature, this temperature is *absolute zero.*

We shall obtain a relation between degrees Kelvin and degrees absolute on the centigrade scale. For this purpose we consider the Carnot cycle of Figure 138 for a perfect gas. The isothermal processes from $D$ to $A$ and from $B$ to $C$ are arcs of equilateral hyperbolas with the equations

$$pV = nR\theta_1,$$
$$pV = nR\theta_2,$$

for absolute temperatures $\theta_1$, $\theta_2$, pressure $p$, volume $V$, gas constant $R$ and number of moles $n$. For the adiabatic processes from $A$ to $B$ and $C$ to $D$ we have the equations

$$pV^\gamma = K_1,$$
$$pV^\gamma = K_2$$

in the adiabatic exponent $\gamma$, and constants $K_1$, $K_2$. By the first law of thermodynamics (see 6.21) we have

$$\delta Q = nc_v d\theta + pdV \tag{7.9}$$

for the temperature $\theta$, heat $\delta Q$, and the specific heat $c_v$ at constant volume. Applying this relation to the isothermal process from $B$ to $C$ we find that the heat $Q_2$ absorbed by the engine is

$$Q_2 = \int_{V_B}^{V_C} p\,dV = nR\theta_2 \ln \frac{V_C}{V_B},$$

where $V_B$ and $V_C$ are the values of $V$ at $B$ and $C$ respectively. Similarly,

$$Q_1 = nR\theta_1 \ln \frac{V_D}{V_A}$$

for the heat $Q_1$ rejected to the cold reservoir during the isothermal process from $D$ to $A$, and the volumes $V_A$ and $V_D$ at $A$ and $D$ respectively. Thus

$$\frac{Q_1}{Q_2} = \frac{\theta_1 \ln \dfrac{V_D}{V_A}}{\theta_2 \ln \dfrac{V_C}{V_B}}.$$

Since the process from $A$ to $B$ is adiabatic we set $\delta Q$ equal to zero in (7.9) to obtain

$$p\,dV = -nc_v\,d\theta,$$

which can be written as

$$R\theta \frac{dV}{V} = -c_v\,d\theta.$$

Integrating from $A$ to $B$ we have

$$\frac{1}{R}\int_{\theta_1}^{\theta_2} c_v \frac{d\theta}{\theta} = \ln \frac{V_A}{V_B}.$$

Similarly for the adiabatic process from $C$ to $D$ we have

$$\frac{1}{R}\int_{\theta_2}^{\theta_1} c_v \frac{d\theta}{\theta} = -\ln \frac{V_D}{V_C}.$$

Therefore

$$\ln \frac{V_A}{V_B} = \ln \frac{V_D}{V_C},$$

Whence

$$\ln \frac{V_C}{V_B} = \ln \frac{V_D}{V_A}.$$

Thus

$$\frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2}.$$

It follows that

$$\frac{\theta_1}{\theta_2} = \frac{T_1}{T_2}$$

for the Kelvin temperatures $T_1$ and $T_2$ associated with the cold and hot reservoirs, respectively.

For the steam and ice points

$$\frac{\theta_s}{\theta_i} = \frac{T_s}{T_i},$$

where the subscript *s* indicates a temperature at the steam point and the subscript *i* indicates a temperature at the ice point. Since the number of degrees between these points is 100 for both the Kelvin and centigrade temperature scales we have

$$\frac{\theta_i + 100}{\theta_i} = \frac{T_i + 100}{T_i}.$$

Thus $\theta_i$ and $T_i$ are equal. Since for any value of $\theta$ the corresponding value of $T$ satisfies the equality

$$\frac{\theta}{\theta_i} = \frac{T}{T_i},$$

we have

$$\theta = T.$$

Thus the *Kelvin temperature is equal to the absolute temperature in degrees centigrade.*

**7.6. Entropy.** With each isothermal we can associate a corresponding temperature. It would also be desirable to be able to associate a value of a convenient variable with each reversible adiabatic process. Such a variable called *entropy* will now be introduced. Except where mention is made to the contrary, in the sections to follow a *reversible process* will mean a directly reversible process, where if we go from state $A$ to state $B$ through intermediate states, we can go through these states from $A$ to $B$ in reverse order, and the process is so slow that the intermediate states are states of equilibrium.

We shall consider curves passing through a region of the $pv$-plane for a given substance. Here $p$ denotes pressure, $v$ specific volume, and $T$ to follow will be Kelvin temperature. We assume that for the region considered two isothermals do not pass through the same point. We know from experience that a substance with an equation of state cannot generally (there are exceptions) be at the same pressure and volume in two different states with different temperatures. Further, that two isothermals do not pass through the same point follows from the assumption that $T$ is a single-valued function of $p$ and $v$, which assumption implies that there is exactly one isothermal curve through a point $(p, v)$ of the $p$, $v$-plane. This assumption underlies the development to follow. As an exception, water at maximum density has intersecting isothermals.

*Through each point* of the part of the $pv$-plane of physical interest *there is a reversible isothermal.* This can be seen as follows. We assign values to $p$ and $v$, and obtain a corresponding value of $T$. We place the substance with this set of values of $p$, $v$, and $T$ in a cylinder whose wall is a reservoir at the temperature $T$. By moving a piston in the cylinder we can make the substance undergo an isothermal process that is clearly reversible, it being assumed that the substance is in thermal equilibrium for each position of the piston, and the work is stored for use in reversing the process. To realize approximately a process in which a substance goes through equilibrium states we can perform the process very slowly.

*Each curve on the pv-plane may be considered to represent a reversible process.* An irreversible process cannot be represented by a curve on this plane.

In practive, *even through a process takes place rapidly,* as in the explosion of a gasoline mixture in the cylinder of a gasoline engine, *we can often attain very accurate results by supposing that the substance undergoing the process is at each instant in thermal equilibrium.*

We can take a substance at a given pressure $p$, specific volume $v$, and temperature $T$, and insulate it from the environment by placing it in a cylinder with a piston. We suppose that the piston is moved in such a way that the work is stored for use in reversing the process, and that the thermodynamic coordinates of the substance are changed adiabatically and reversibly. *There is thus a reversible adiabatic through each point* of physical interest on the $pv$-plane of the substance. Of course, in practice we can only approximate reversible adiabatics.

Suppose that an arc $C$ of a reversible adiabatic and an isothermal coincide. We exclude $T = 0$, because by equation (7.15) to follow, the vanishing of $T$ along a curve in the $p$, $v$-plane implies the vanishing of the heat $\Delta Q$ transferred. Along an isothermal we have $T = T_0$ for a constant $T_0$. We assume that we can solve the equation of state $T_0 = T(p, v)$ for $p$ in terms of $v$. Substituting for $p$ in (6.10), assuming that the differentiations can be performed, and setting $\delta Q$ equal to zero, we obtain a differential relation in $v$, which we suppose is $\not\equiv 0$ in $dv$. It follows that $v$ is a constant. We know, however, that we can vary the volume of a substance adiabatically, a contradiction. We might have solved $T_0 = T(p, v)$ for $v$ in terms of $p$, and gotten $p$ a constant, again a contradiction. Hence the *reversible adiabatic and isothermal arcs do not coincide.* We also conclude that *adiabatic and isothermal curves are not asymptotic.*

Suppose that two reversible adiabatics $A_1$ and $A_2$ cross at a point $P$. Let $M$ be a point on $A_1$ distinct from $P$ and not on $A_2$, and let $I_M$ be the isothermal through $M$. Suppose that $I_M$ meets the adiabatic $A_2$ in a point. The curves $A_1$, $A_2$, and $I_M$ now form a loop composed of an adiabatic and an isothermal. We describe the loop clockwise with an engine $E$. The engine then must draw some heat $Q$ from a reservoir (during $I_M$) and turn this directly into work. This violates the second law of thermodynamics. Thus we cannot have a loop formed from two reversible adiabatics and a reversible isothermal.

Suppose, now, that regardless of the position of $M$ on $A_1$ the isothermal

$I_M$ does not meet the adiabatic $A_2$. As the point $M$ crosses $P$ the isothermal $I_M$ deforms continuously, by assumption, from what it is on one side of $A_2$ to what it is on the other side. This implies that $A_2$ is an isothermal, a contradiction. In any case *adiabatic curves do not cross.*

The assumption that $I_M$ deforms continuously as $M$ moves is based on experience. The concept of continuous deformation can be expressed in precise mathematical terms, and a rigorous treatment given.

By the discussion above concerning a loop composed of an adiabatic and isothermal we know that *a reversible adiabatic curve can cut a reversible isothermal in at most one point.*

Suppose that a reversible isothermal $I$ and a reversible adiabatic $A$ do not cross. We select a continuous curve $CDE$ that crosses $A$ at the point $D$ between $C$ and $E$, and of which the one end $C$ is on $I$. It is assumed that this curve is of finite length. Let $F$ be a point on this curve. As $F$ moves from $C$ to $D$ the isothermal through $F$ deforms continuously into an isothermal $I_D$ that is tangent to the adiabatic $A$ at $D$. We are assuming that the adiabatics and isothermals have tangents at each point, and that as we move along any continuous curve the isothermal through a point of the curve deforms continuously. We can move a point $F'$ on the curve $CDE$ from the point $E$ to $D$. The corresponding isothermal will deform continuously into the isothermal through $D$. This isothermal will not cut $A$ since $I$ does not cut $A$. Thus approaching the point $D$ from either side on the curve $CDE$, the isothermals do not cut $A$, which is between them, but still approach each other. Then $A$ is the limiting isothermal at $D$, a contradiction. We conclude that *each reversible isothermal cuts each reversible adiabatic.*

In any region of the $pv$-plane with which we may be concerned it is assumed that an adiabatic or isothermal cannot end abruptly at a point within the region, but must *go from boundary to boundary.* If we are concerned with an unbounded portion of space, we suppose that for every region belonging to this portion of space the adiabatic and isothermal curves go from boundary to boundary, or recede from any finite portion of the plane.

The discussion above shows that *the adiabatic and isothermal families of curves of a given substance have convenient properties.*

We consider an arc $AB$ of the $pv$-plane of a substance, where $AB$ joins adiabatics $A_1$ and $A_2$ through $A$ and $B$, respectively, and the arc $AB$ lies between the curves $A_1$ and $A_2$ (see Figure 139). We can find isothermals $I_1$ and $I_2$ so that the region bounded by the adiabatics $A_1$ and $A_2$ and the isothermals $I_1$ and $I_2$ contains the arc $AB$. We can then find an intermediate isothermal $I$ so that the area (by integration) under the curve $ACDB$, composed of the adiabatic arcs $AC$ and $DB$, and the isothermal arc $CD$ of $I$ joining $A_1$ and $A_2$, is equal to the area under the given arc $AB$ (these areas are of regions extending to the $v$-axis).

By (6.7) we have

$$Q_{AB} = U_B - U_A + W_{AB}$$

for the heat $Q_{AB}$ transferred during the process $AB$, the internal energy $U_B$ at $B$, the corresponding energy $U_A$ at $A$, and the work $W_{AB}$ for the

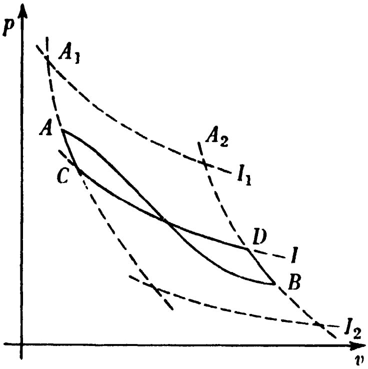

Fig. 139. Arc $AB$ joining adiabatics $A_1$ and $A_2$.

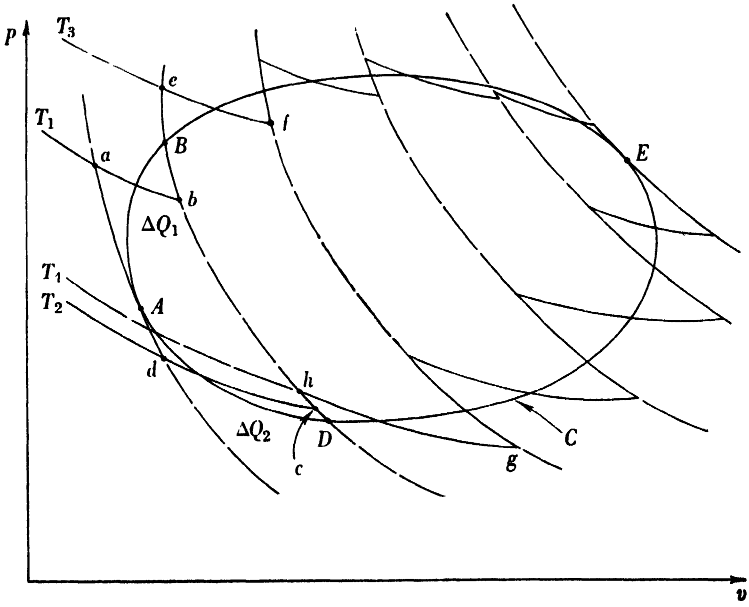

Fig. 140. Closed reversible process $C$.

process $AB$. Since $U_A$ and $U_B$ do not depend on the curve joining $A$ and $B$, and the work for $ACDB$ is $W_{AB}$, the heat transferred during the process $ACDB$ is $Q_{AB}$, whence the heat transferred during the isothermal portion $CD$ is equal to the heat transferred during the original process $AB$.

Let us consider a simply connected region on the $pv$-plane bounded by a continuous curve $C$. We assume that each arc of $C$ represents a reversible process. See Figure 140. Through a point in the region bounded by $C$ we can draw an adiabatic $A_1$. Through a point of the region not on $A_1$ we draw a second adiabatic $A_2$. We shall assume that in the portion of the plane between $A_1$ and $A_2$ there are only a finite number of arcs of the curve $C$ cut out by these adiabatics. We assume that $C$ is of finite length. We can then pass between $A_1$ and $A_2$ adiabatics that subdivide the arcs between $A_1$ and $A_2$ into subarcs of maximum length $\delta$, where $\delta$ is an arbitrary positive number. We can similarly subdivide the rest of the curve $C$ by adiabatics so that no arc of $C$ so obtained has length greater than $\delta$. In Figure 140 we have pictured the adiabatics by dashed curves running from the upper part of the first quadrant to the lower part.

We let $A$ and $E$ be points at which adiabatics meet the curve $C$ but do not cross $C$, so that the adiabatics that cross $C$ lie between the adiabatics through $A$ and $E$. Consider a set of adiabatics that cut the curve $C$ into a finite number of arcs of maximum length $\delta$, where this set includes the adiabatics through $A$ and $E$. Let the curve and adiabatics be such that the arcs can be paired so that we can associate a Carnot cycle with each such pair of arcs in the following manner. In Figure 140 we have pictured such a pair of arcs $AB$ and $AD$. The arc $AB$ is replaced by $AabB$, where $Aa$ and $bB$ are adiabatic arcs, and $ab$ is an isothermal arc such that the heat $\Delta Q_1$ transferred during the process $ab$ is equal to the heat transferred during the process $AB$. Similarly, the heat $\Delta Q_2$ transferred during the process $AD$ is equal to the heat transferred during the process corresponding to the isothermal arc $dc$. The process corresponding to the closed loop $abcd$ is thus a Carnot cycle. The loop $efgh$ for the two arcs adjoining $AB$ and $AD$ is another Carnot cycle.

We shall use notations $\Delta Q_1$ and $\Delta Q_2$ to designate signed quantities. Here $\Delta Q_1$ is positive because this is heat taken on by the engine, and $\Delta Q_2$ is negative, being heat rejected to a cold reservoir. As usual $T$'s denote degrees Kelvin. By Section 7.5

$$\frac{\Delta Q_1}{T_1} + \frac{\Delta Q_2}{T_2} = 0.$$

Similarly, for the cycle $efgh$ we have an equation

$$\frac{\Delta Q_3}{T_3} + \frac{\Delta Q_4}{T_4} = 0.$$

If there are $n$ cycles in all, the equations of this type may be added to give

$$\sum_{i=1}^{2n} \frac{\Delta Q_i}{T_i} = 0. \tag{7.10}$$

After a relabeling of the subscripts we can write the sum in (7.10) as a sum of two summations

$$\sum_{j=1}^{m} \frac{\Delta Q_j}{T_j} + \sum_{k=1}^{2n-m} \frac{\Delta Q_k}{T_k},$$

where the first involves heat increments for the arc $ABE$ only, and the second increments for the arc $EDA$. We suppose that we are working with a unit weight, say one pound of the given substance. If we take the limit of the first summation as $\delta$ goes to zero, assuming that this limit exists, we obtain the line integral

$$\int_{ABE} \frac{du + pdv}{T}$$

along the curve $ABE$ for the internal energy $u$ (per pound) and specific volume $v$, whereas the limit of the second summation as $\delta$ goes to zero is the integral

$$\int_{EDA} \frac{du + pdv}{T}$$

along the arc $EDA$, if this integral exists. The sum of these integrals is then the limit of the left member of (7.10) as $\delta$ goes to zero. Since this limit is zero we have (for the clockwise integral around $C$)

$$\int_C \frac{du + pdv}{T} = \int_{ABE} \frac{du + pdv}{T} + \int_{EDA} \frac{du + pdv}{T} = 0. \qquad (7.11)$$

Let us assume that the curve $C$ represents a directly reversible process bringing a body from the state represented by the point $A$ back to the point $A$. The process can then be reversed so that we can run through the states in counterclockwise order as well as clockwise. Since the line integral of $(du + pdv)/T$ along $EDA$ is the negative of the line integral along $ADE$ we have

$$\int_{ABE} \frac{du + pdv}{T} = \int_{ADE} \frac{du + pdv}{T}. \qquad (7.12)$$

The points $A$ and $E$ were specially chosen on the curve $C$. Since the line integral of $(du + pdv)/T$ around $C$ vanishes, we have (7.12) valid for any choice of points $A$ and $E$ on $C$, assuming that the integrals exist. Since (7.12) is valid for every curve $C$ on which $A$ and $E$ lie, where $C$ has the properties assumed in the above argument, we conclude that the value of the line integral

$$\int_A^E \frac{du + pdv}{T} \qquad (7.13)$$

is independent of the path joining the points $A$ and $E$ for a large class of paths. We decide that the differential expression

$$\frac{du + pdv}{T} \tag{7.14}$$

is exact and is the total differential $ds$ of a function $s$.

We can now state a mathematical consequence of the second law of thermodynamics.

*The expression (7.14) is the total differential $ds$ of a function $s$ of two of the variables in the set $p$, $v$, $T$ (the function being called the entropy of the given substance).*

Experience shows that the entropy function exists even for those exceptional cases where isothermals cross each other. The above result of the second law of thermodynamics was proved by assuming reasonable properties concerning adiabatics, isothermals, and limits. We postulate that an entropy function exists for each thermodynamic system. The variables need not be $p$, $v$, and $T$, but may, for example, be the tension in a wire, the length of the wire, and the absolute temperature, it being assumed that the pressure and volume changes in the wire are negligible.

The integral (7.13) is $(s_E - s_A)$ for the entropy $s_E$ at $E$ and entropy $s_A$ at $A$. We can start with any state $A$ as a reference state and choose an arbitrary real value for $s_A$. The integral (7.13) now determines the entropy at $E$. Suppose that this integral is over a directly reversible process. Then having determined the value of the entropy function at the point $E$ we can return to the point $A$ so that the state of the given substance and its environment that first existed at $A$ is revived. We can now leave $A$ along a reversible path to a point $F$ on the $pv$-plane, and determine the entropy at this point. We can then return to $A$ and the initial state at $A$, and then embark along a reversible path to another point of the $pv$-plane. In this manner we can determine values of the entropy function for the given substance at all points of interest in the region of the $pv$-plane associated with this substance.

If the curve joining $A$ to $E$ is a reversible adiabatic, we have

$$du + pdv = 0,$$

so that the integral (7.13) vanishes. It follows that *along a reversible adiabatic the entropy is constant.* The reversible adiabatic is therefore an *isentropic.* With distinct reversible adiabatics go different values of the entropy.

Along an isothermal the temperature $T$ is a constant. Therefore *if $Q$ units of heat are transferred to a substance during a reversible isothermal process with the temperature $T$, the change in entropy of the substance is $Q/T$.*

In the statement and proof of Law 7.3 to follow the term "*reversible*" will mean only that the system and its surroundings can be restored to their original state, whereas "*directly reversible*" will mean that we are always in equilibrium and can traverse the intermediate states in reverse order.

The second law of thermodynamics can be stated in the following form.

*Law* **7.3. Mathematical formulation of the second law of thermodynamics.** *Consider a thermodynamic system. There exists an integrating factor $T$ such that*

$$\frac{\delta Q}{T}$$

*is the total differential $ds$ of a function (entropy) $s$ associated with the given system, and for an adiabatic process the change in entropy $\Delta s$ is positive or zero, while for a reversible adiabatic process $\Delta s = 0$.*

For a directly reversible adiabatic process $\delta Q = 0$, so that the entropy $s$ is constant, whence $\Delta s = 0$.

We assume that $p$ and $v$ are single valued functions of $T$ and $s$ for all values of $T$ and $s$ which enter into the argument, and that along directly reversible isothermals $s$ is a single valued function of $p$ or $v$, and along directly reversible adiabatics $T$ is a single valued function of $p$ or $v$.

Suppose now that we go from a state $A$ with the temperature $T_1$ and entropy $s_1$ to a state $B$ with the temperature $T_2$ and entropy $s_2$ by an adiabatic process $AB$. We can take the given system from $B$ back to $A$ by the following reversible process. Let the system undergo a reversible adiabatic process $BC$ to some arbitrary temperature $T_3$. Then subject the system to a reversible isothermal process $CD$ until at some state $D$ the reversible adiabatic with entropy $s_1$ is reached. Now let the system undergo a reversible adiabatic process $DA$. We have thus a cycle composed of the adiabatic processes $AB$, $BC$, and $DA$, and the isothermal process $CD$. Let $Q$ be the heat received or lost by the system and $W$ the work done during the cycle. By the first law of thermodynamics

$$Q = W.$$

If $W$ and $Q$ were positive, the system would receive heat and convert this into work. We may suppose that there are no other thermodynamic effects, and that the system contacts a reservoir during $CD$. Law 7.1 is now violated. Thus $Q \leqq 0$. Let $\Delta s$ be the difference $s_2 - s_1$. For $CD$ we have

$$Q = T_3(s_1 - s_2),$$

whence

$$\Delta s \geqq 0.$$

If $Q$ is zero we have

$$W = \Delta s = 0,$$

and the process $AB$ is a reversible adiabatic whenever the environment is unaffected by the process $ABCDA$. The process $BCDA$ restores the original state of the system.

Suppose that the adiabatic process $AB$ is reversible. We can then go from $B$ back to $A$ by a process $BA$, restoring the original state of the system and its environment. For the process $BA$ there is then no net heat transfer between the system and its surroundings. Consider the cycle composed of the path $ADCB$ (the reverse of $BCDA$ described above) and the process $BA$. For this cycle the work $W$ is done, where

$$W = T_3(s_2 - s_1) = T_3\Delta s.$$

If $\Delta s > 0$, this violates Law 7.1. It follows that $\Delta s = 0$ for a reversible adiabatic process.

Law 7.3 is thus a consequence of Laws 7.1 and 7.2.

Through friction, work can be turned directly into an equivalent amount of heat. Thus $Q$ and $W$ above can both be negative.

To prove that Law 7.1 follows from Law 7.3 note that if heat $-Q$ is extracted from a reservoir at the absolute temperature $T$ the change in total entropy for this reservoir is $-Q/T$. Since $Q > 0$, it follows that $\Delta s < 0$ for the process described in Law 7.1.

To show that Law 7.2 follows directly from Law 7.3 let $T_1$ and $T_2$ be the absolute temperatures of the cooler and hotter bodies respectively. Let $Q$ be the heat extracted from the cooler body in the process described in Law 7.2. The change in total entropy of the cooler body is then $-Q/T_1$, and this change for the hotter body is $Q/T_2$, so that the total change $\Delta s$ in the entropy of the system (the entropy is defined to be additive) is

$$Q\left(\frac{1}{T_2} - \frac{1}{T_1}\right)$$

Since $\Delta s < 0$ for this process, Law 7.2 follows from Law 7.3.

The equivalence of Laws 7.1, 7.2, and 7.3 has now been established.

Consider a path on the $pv$-plane. By the theorem of the mean for integrals we can write

$$\Delta Q = T\Delta s \tag{7.15}$$

where $\Delta Q$ is the heat transferred, $\Delta s$ is the change in entropy, and $T$ is a properly chosen value of the temperature between the maximum and minimum taken on along the given path. Consider a reversible path $ABC$ of the $pv$-plane. We can subdivide the curve $ABC$ into arcs, and form the sum of the heats transferred for each arc and written as $\Delta Q$ in (7.15). Taking the limit of this sum (if it exists) as the number of arcs is increased beyond all bounds and the maximum length of the arcs goes to zero, we obtain the heat $Q$ transferred during the process $ABC$, where

$$Q = \int_{ABC} T ds. \tag{7.16}$$

On a $Ts$-diagram $Q$ is clearly the area *under* the curve corresponding to $ABC$ (if each vertical line meets $ABC$ in at most one point). *For a simply closed curve on the Ts-plane the area bounded by this curve is the net heat* *transferred during the process*, which heat is the work $W$. This is analogous to the area representing work $W$ for such a curve on the $pv$-plane.

In Figure 141 we have pictured a reversible cycle. The top part of the curve is designated by $C_1$, and the bottom by $C_2$. These parts meet at the points $A$ and $B$. The area under the curve $C_1$ is the heat $Q_1$ received by the device, whereas the area under $C_2$ is the heat $Q_2$ rejected. The area between $C_1$ and $C_2$ is $(Q_1 - Q_2)$. Since the efficiency of the device is $(Q_1 - Q_2)/Q_1$, it is a simple matter to obtain this efficiency from the figure with a planimeter. This is one reason why the use of entropy is important to engineers.

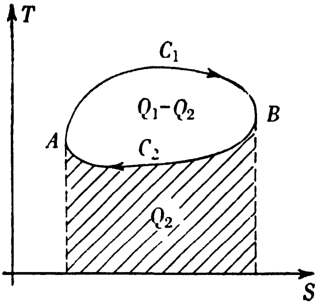

Fig. 141. A reversible cycle.

Tables of entropies of various substances are available, as for air and steam. The entropy may be given for one pound of the substance. In the case of a perfect gas the formulas (6.23), (7.15), and $pV = nRT$ yield the entropy $S$ for $n$ moles of the gas, where

$$S = nc_p \ln T - nR \ln p + C$$

for an arbitrary constant $C$, provided the specific heat $c_p$ is considered constant.

**7.7. Phase transitions.** Changes of phase of a *pure* substance, such as melting and vaporization, occur at constant temperature and pressure. Associated with the transition from a phase $B$ to a phase $A$ there is a change of volume accompanied by a transfer of heat (latent heat). Assuming that the transition takes place reversibly, it being isothermal, the *latent heat* $l$ satisfies

$$l = T(s_A - s_B), \tag{7.17}$$

where $s_A$ is the entropy for the phase $A$, and $s_B$ is the entropy for the phase $B$, the temperature being $T$.

The derivatives which occur in this and the next section are assumed to exist.

We introduce the *Gibbs' function* (often called thermodynamic potential) $g$ given by

$$g = h - Ts,$$

where $h$ is enthalpy and $s$ is entropy, so that

$$dg = dh - Tds - sdT.$$

Since

$$dh = \delta Q + vdp = Tds + vdp,$$

we can write

$$dg = vdp - sdT.$$

Evidently,

$$v = \left(\frac{\partial g}{\partial p}\right)_T, \quad s = -\left(\frac{\partial g}{\partial T}\right)_p. \tag{7.18}$$

For a reversible, isothermal, isobaric process $dg = 0$, and $g$ is a constant.

A *change of phase of the first order* is one where the first partial derivatives of $g$, that is, $v$ and $-s$, change discontinuously.

Since

$$ds = \left(\frac{\partial s}{\partial T}\right)_v dT + \left(\frac{\partial s}{\partial v}\right)_T dv$$

and

$$c_v = T\left(\frac{\partial s}{\partial T}\right)_v,$$

we can write

$$Tds = c_v dT + T\left(\frac{\partial s}{\partial v}\right)_T dv.$$

We introduce the *Helmholz function* $a$ given by

$$a = u - Ts.$$

Since

$$da = du - Tds - sdT,$$

we have

$$da = -pdv - sdT.$$

The equality

$$\frac{\partial}{\partial v}\left(\frac{\partial a}{\partial T}\right) = \frac{\partial}{\partial T}\left(\frac{\partial a}{\partial v}\right)$$

implies that

$$\left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial s}{\partial v}\right)_T.$$

Thus

$$Tds = c_v dT + T\left(\frac{\partial p}{\partial T}\right)_v dv.$$

The following relation in increments is then approximately valid:

$$T\Delta s = c_v \Delta T + T\left(\frac{\partial p}{\partial T}\right)_v \Delta v.$$

For the transition from the phase $B$ to the phase $A$ this gives

$$T(s_A - s_B) = T\left(\frac{\partial p}{\partial T}\right)_v (v_A - v_B).$$

It is customary to write this equation in the form

$$\frac{dp}{dT} = \frac{l}{T(v_A - v_B)}, \tag{7.19}$$

with the aid of (7.17). Equation (7.19) is the *Clapeyron-Clausius equation.*

Suppose, as an example, that water evaporates into steam. It is assumed that the temperature, pressure, and specific volumes of both phases are given, and that the vapor pressure curves are known so that we can obtain $dp/dT$. Equation (7.19) now gives the latent heat $l$.

The enthalpy and entropy of water are generally based on assigning the value zero to the state where the water is at 32 degrees Fahrenheit and the pressure is 0.0886 pounds per square inch. This pressure is the one at which water evaporates for the temperature of 32 degrees Fahrenheit.

Since

$$Tds = dh - vdp, \tag{7.20}$$

and $dp = 0$ for the transition from water to steam, we have

$$Tds = dh,$$

so that for the enthalpies $h_A$ and $h_B$ associated with phases $A$ and $B$, respectively,

$$T(s_A - s_B) = h_A - h_B.$$

We can thus obtain the enthalpy of the steam by adding latent heat to the enthalpy of the water.

## 7.8. Entropy of a mixture of perfect gases.

We suppose that we have a mixture of $r$ perfect gases, relatively inert to each other. Let $n_i$ denote the number of moles of the $i$th gas for $i = 1, \ldots, r$. Let $U$, $V$, $T$, $p$, and $S$ designate, respectively, the internal energy, volume, temperature, pressure, and entropy of the mixture, and let $u_i$, $p_i$, $v_i$, and $s_i$ designate the internal energy density, partial pressure, specific volume, and entropy of the $i$th gas. Since

$$n_iv_i = V, \qquad p = p_1 + p_2 + \cdots + p_r,$$

we have

$$dS = \frac{dU + pdV}{T} = \frac{\sum_{i=1}^{r} n_idu_i + \sum_{i=1}^{r} p_id(n_iv_i)}{T}.$$

Thus

$$dS = \sum_{i=1}^{r} n_ids_i,$$

where

$$ds_i = \frac{du_i + p_i dv_i}{T}.$$

Let $S_i$ be the total entropy of the $i$th gas, so that

$$S_i = n_i s_i.$$

It follows that

$$dS = d\left(\sum_{i=1}^{r} S_i\right).$$

Thus $S$ differs from the sum of the $S_i$'s by a constant, which we may take to be zero. Then *the entropy of a mixture of perfect gases*, relatively inert to each other, *is the sum of the entropies of these gases.*

**7.9. Adiabatic equation of state.** By (7.20)

$$ds = \frac{dh - v dp}{T}$$

for the entropy $s$, enthalpy $h$, specific volume $v$, pressure $p$, and temperature $T$ of a substance. From

$$dh = \frac{\partial h}{\partial T}\,dT + \frac{\partial h}{\partial p}\,dp,$$

we have

$$ds = \frac{\left(\frac{\partial h}{\partial T}\right)_p dT + \left[\left(\frac{\partial h}{\partial p}\right)_T - v\right] dp}{T}.$$

For a reversible adiabatic process $ds = 0$. Thus

$$\left(\frac{\partial h}{\partial T}\right)_p dT + \left[\left(\frac{\partial h}{\partial p}\right)_T - v\right] dp = 0. \tag{7.21}$$

From (7.20)

$$\left(\frac{\partial h}{\partial p}\right)_T = T\left(\frac{\partial s}{\partial p}\right)_T + v. \tag{7.22}$$

By (7.18) and

$$\frac{\partial}{\partial T}\left(\frac{\partial g}{\partial p}\right) = \frac{\partial}{\partial p}\left(\frac{\partial g}{\partial T}\right)$$

there follows

$$\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial s}{\partial p}\right)_T. \tag{7.23}$$

In view of (7.22) and (7.23)

$$\left(\frac{\partial h}{\partial p}\right)_T - v = -T\left(\frac{\partial v}{\partial T}\right)_p.$$

From this relation and

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p$$

equation (7.21) takes the form

$$c_p dT - T\left(\frac{\partial v}{\partial T}\right)_p dp = 0. \tag{7.24}$$

From the equation of state for $h$ in terms of $p$ and $T$, the constant $c_p$ can be obtained, at least theoretically, as a function of $p$ and $T$. If we have an equation of state expressing $v$ as a function of $p$ and $T$, we can, again at least theoretically, integrate (7.24) under rather general conditions to obtain $T$ as a function of $p$. Substituting this function for $T$ in $v$, the adiabatic relation

$$v = v(p)$$

between $v$ and $p$ is attained.

The treatment of this section is valid for any substance, and is not restricted to perfect gases.

**7.10. Refrigeration cycle.** We shall describe a simple refrigeration cycle pictured on a $Ts$-diagram. In Figure 142 the point $A$ corresponds to the refrigerant at the point of evaporation. The line segment $AB$ corresponds to the reversible isothermal, isobaric vaporization of the refrigerant at the temperature $T_1$. During the evaporation of the refrigerant, heat is extracted from the body to be cooled. This body might be the interior of a refrigerator. The vertical line segment $BC$ corresponds to the isentropic compression of the refrigerant in the compressor until the temperature $T_2$ is reached. The curve $CD$ indicates reversible isobaric cooling of the refrigerator vapor by rejection of heat to a hot body, such as through the coils of a refrigerator to the air of a room. The refrigerant reaches the condensation temperature $T_3$ at $D$. The horizontal line segment $DE$ represents reversible isothermal isobaric condensation of the vapor to a liquid. The whole process $CE$ occurs in the condenser, from which the liquid proceeds to a storage reservoir. The dashed line $EA$ indicates an irreversible throttling process through a valve, during which the pressure and temperature of the refrigerant drop.

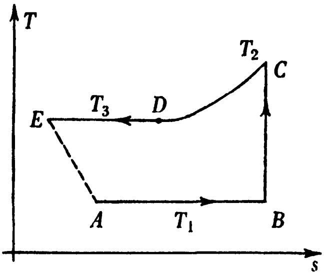

Fig. 142. Refrigeration cycle.

**7.11. Steam flow through a nozzle.** In this section we shall show how we treat a problem involving a gas not assumed to be a perfect one.

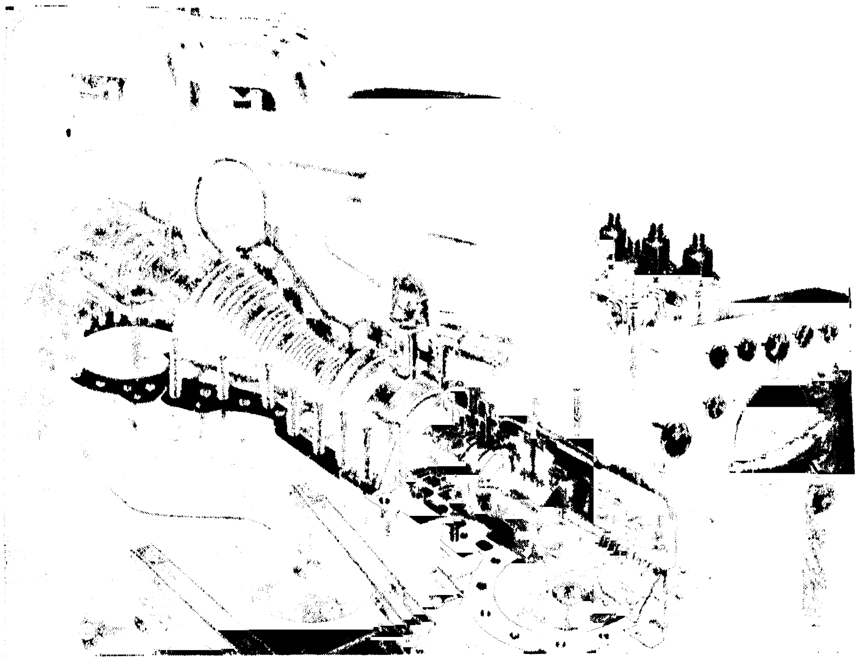

Fig. 143. Cutaway of a 3,600 rpm condensing impulse reaction steam turbine. Courtesy of the Allis-Chalmers Manufacturing Company.

Let $p_1$ be the inlet pressure to a nozzle, $p_2$ the back pressure, and $w$ the weight rate of flow of steam through the nozzle. Suppose that $p_1 = 100$ pounds per square inch, that $p_2$ is one atmosphere, and that the steam ahead of the nozzle is at 418 degrees Fahrenheit (see Figure 144). For *superheated* steam, with which we are working, the critical pressure is approximately 0.55 times $p_1$. Since the critical pressure in our example is 55 pounds per square inch and atmospheric pressure is less than this, we must take $p_2 = 55$.

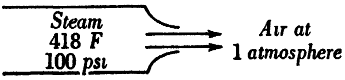

Fig. 144. Steam flow through a nozzle.

From tables of Keenan and Keyes (see reference 4 at the end of this chapter) the entropy of the steam at 418 degrees Fahrenheit and 100 pounds per square inch pressure is approximately 1.663 British thermal units per pound per degree Fahrenheit and the enthalpy is 1237 British thermal units per pound. Assuming that we are working with an isentropic, the enthalpy at the exit of the nozzle (corresponding to 55 pounds per square inch and entropy 1.663 British thermal units per pound per degree Fahrenheit) is 1185 British thermal units per pound. The enthalpy difference is 52 British thermal units per pound, which multiplied by Joule's constant $J$ gives $52J$ foot-pounds per pound. From (6.15) we have

$$\frac{u^2}{2g} = 52J$$

for the magnitude $g$ of the acceleration of gravity, and the emerging speed $u$ of the steam.

In this example the steam is superheated, containing no water vapor. If we have such a mixture we need only use the entropy of the mixture; this can be found from tables based on experiment.

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## CHAPTER

## 8

# *Heat transfer*

**8.1. Conduction.** In the study of thermodynamics we have concerned ourselves with the amounts of energy transformed in various processes, and transferred as heat from one body to another. In the subject of heat transfer we are concerned with the actual methods by which energy in the form of heat is passed from substance to substance. From physical considerations we know that heat transfer takes place through conduction, convection, and radiation.

Heat conduction is passage of heat energy through a body resulting from molecular motion, and not due to rays or the relative movement of parts of the body. Suppose that we have an infinite homogeneous slab of thickness $d$. Suppose further that the solid is in steady state with the temperature $\theta_1$ on one face $F_1$ and temperature $\theta_2$ on the other face $F_2$, where $\theta_1 > \theta_2$. Consider a region $R$ of area $A$ on the face $F_1$. We let $dQ/dt$ be the rate of flow of heat with respect to time across $R$. We would expect $dQ/dt$ to decrease with an increase of $d$, to increase with $A$, and to increase with $(\theta_1 - \theta_2)$. The flow of heat is in the direction of decreasing temperature. The simplest formula expressing these properties is

$$\frac{dQ}{dt} = kA\frac{(\theta_1 - \theta_2)}{d}$$

for a constant $k$. Experiments show this law to be valid. An obvious generalization of this law is the following.

***Law* 8.1. Fourier's law.** *Let R be a plane region of area A, on or in a homogeneous substance, and let x be a Cartesian coordinate along a normal to R. Suppose that the derivative $d\theta/dx$ of temperature $\theta$ evaluated along a normal to R is the same at each point of R. The rate $dQ/dt$ of flow of heat across R due to conduction in the direction of increasing x is given by*

$$\frac{dQ}{dt} = -kA\frac{d\theta}{dx} \tag{8.1}$$

*for a constant k.*

The constant $k$ is termed the *thermal conductivity* of the substance.

If one of the assumptions of Law 8.1 is not satisfied, as when the conductivity of the material varies over the region $R$, or $R$ is not a plane region, we can extend (8.1) to these situations by integration to obtain

$$\frac{dQ}{dt} = -\iint_R k\frac{d\theta}{dn}\,d\Sigma \tag{8.2}$$

for the normal derivative $d\theta/dn$.

With $k$ in British thermal units per hour per foot per degree Fahrenheit we have the following experimental values. For 99.9 per cent pure copper at 32 degrees Fahrenheit, $k$ is 220; at 392 degrees, $k = 215$, and at 1112 degrees, $k = 206$. For iron of the same purity the corresponding values are 43, 36, and 22. For aluminum 99.7 per cent pure, $k$ has approximately the value 132 for the entire range from $-60$ to 700 degrees Fahrenheit.

For water at 32 degrees Fahrenheit we have $k = 0.330$, whereas at 140 degrees the value is 0.381. For air at 32 degrees Fahrenheit and 1 atmosphere, $k = 0.0140$.

For glass $k$ varies from 0.333 to 0.5. For kapok at 86 degrees Fahrenheit with a density of 0.88 pounds per cubic foot we have $k = 0.020$, and for asbestos at 500 degrees Fahrenheit with a density of 36 pounds per cubic foot the value of $k$ is 0.122.

The range of $k$ is quite large, being almost zero for gases under great vacuum, and as much as 7,000 for copper at $-422$ degrees Fahrenheit.

**8.2. Heat stored.** From the definition of specific heat in Section 6.4 it follows that the rate $dQ/dt$ at which heat is stored in a three-dimensional region $R$ of a substance is

$$\frac{dQ}{dt} = wc_pV\frac{\partial\theta}{\partial t} \tag{8.3}$$

for the specific weight $w$ of the substance, specific heat $c_p$, volume $V$ of the region $R$, and rate $\partial\theta/\partial t$ of change of the temperature $\theta$ of the substance, it being assumed that the same state prevails at each point of $R$.

If the states at different points of $R$ vary, we must replace the right member of (8.3) by an integral to obtain

$$\frac{dQ}{dt} = \iiint_R wc_p\frac{\partial\theta}{\partial t}\,dV. \tag{8.4}$$

**8.3. Heat transfer through a wall.** Consider a homogeneous panel of uniform thickness, such as the door of a room, and let us suppose that the panel is initially at a temperature $\theta_1$, such as the temperature of the room. The panel is suddenly exposed to a temperature $\theta_2$ on one side, the temperature on the other side being kept at $\theta_1$. This happens, for example, if the door of a room leading outdoors is closed by swinging the door from a position inside the room to the closed position, $\theta_2$ being the outside temperature. It is desired to describe the transient condition in the panel.

In practice the temperature of a door surface will not be the temperature of the air to which the surface is exposed. To simplify the exposition, we shall, however, neglect this difference until a later section.

The thickness of a panel is generally small compared to its other dimensions, whence we replace the panel in our mathematical treatment by an infinite slab. Let the thickness of the slab be designated by $D$. Suppose that $\theta_1$ is higher than $\theta_2$. We let the side at the temperature $\theta_1$ be denoted by $S_1$ and the other side by $S_2$. We let $x$ be the coordinate of distance measured perpendicular to $S_1$, so that $S_1$ has the coordinate $x = 0$ and $S_2$ has the coordinate $x = D$.

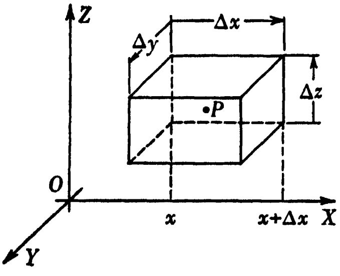

Fig. 145. Element of a panel.

We consider a point $P$ in or on the panel, and take an element of volume containing $P$ as shown in Figure 145. Cartesian coordinates $y$ and $z$ are introduced so that the volume of the element is $\Delta x \Delta y \Delta z$. Let the points on one face of the element have the coordinate $x$ and on the opposite face the coordinate $(x + \Delta x)$. Because of uniformity, the heat flow into and out of the element is through these faces only. The temperature $\theta$ at points with the coordinate $x$ is a function $\theta(x, t)$ of both $x$ and $t$.

By (8.1) the net rate $g_c$ of heat flow across the faces of the element is given by

$$g_c = -k\Delta y\Delta z \left[\frac{\partial\theta}{\partial x}\right]_x + k\Delta y\Delta z \left[\frac{\partial\theta}{\partial x}\right]_{x+\Delta x}.$$

Dropping an $\epsilon$-term (not to be confused with emissivity) which does not contribute to the final limit, we have ($\partial^2\theta/\partial x^2$ is assumed to exist)

$$g_c = k\Delta y\Delta z \frac{\partial^2\theta}{\partial x^2}\Delta x.$$

By (8.3) the rate $g_s$ at which heat is stored in the element is given by

$$g_s = wc_p\Delta x\Delta y\Delta z \frac{\partial\theta}{\partial t}$$

for the specific weight $w$ and specific heat $c_p$ of the panel, where $\partial\theta/\partial t$ is evaluated at a properly chosen point of the element. Since $q_c = q_s$ for each element, taking the limit as the element shrinks to $P$, we have

$$\frac{\partial\theta}{\partial t} = \alpha \frac{\partial^2\theta}{\partial x^2} \tag{8.5}$$

valid at $P$, where $\alpha$ is the *thermal diffusivity* $k/wc_p$.

We have the boundary conditions

$$x = 0, \quad \theta = \theta_1; \qquad x = D, \quad \theta = \theta_2. \tag{8.6}$$

The initial and final conditions are:

$$\begin{aligned} t &= 0, \quad \theta = \theta_1, \quad 0 \leqq x \leqq D. \\ t &\to \infty, \theta \to \theta_1 - \frac{x}{D}(\theta_1 - \theta_2), \quad 0 \leqq x \leqq D. \end{aligned} \tag{8.7}$$

The system (8.5)–(8.7) describes the conditions in the panel.

**8.4. Convection.** Heat convection is the transfer of heat between moving parts of a fluid, or between a fluid and a surface moving relative to each other. In the case of the door of Section 8.3 we have convection of heat from the air to the door at $x = 0$.

***Law* 8.2. Newton's law of heat transfer by convection.** *Suppose that a surface of area $A$ is exposed to a fluid, and that a uniform temperature difference $\Delta\theta$ exists between the fluid at the surface and the other material with which it is in contact. The rate $dQ/dt$ of transfer of heat satisfies the law*

$$\frac{dQ}{dt} = hA\Delta\theta \tag{8.8}$$

*for a constant $h$.*

This law is an experimental one in common use. The constant $h$ is called the *film coefficient of heat transfer*, or the *coefficient of heat transfer.* It is often convenient to add to the coefficient of heat transfer a correction to take care of radiation, and thus obtain a *combined radiation and convection coefficient* of heat transfer.

If there are not uniform conditions at points of the given surface, we must replace the right member of (8.8) by an integral, to obtain

$$\frac{dQ}{dt} = \iint h\Delta\theta d\Sigma \tag{8.9}$$

where the integration is over the given surface, the quantities $h$ and $\Delta\theta$ being functions of a point on the surface.

Numerous values of $h$ have been determined experimentally, and can be found in tables. In what follows $h$ is in British thermal units per hour per square foot per degree Fahrenheit temperature difference. In the case of natural convection where the flow of the fluid is due to the contact at the given surface (and not, for example, winds or forced flow) we have the following, where the fluid is air at one atmosphere and ordinary temperatures such as 100 degrees Fahrenheit. For vertical plates more than 3 feet high we have $h = 0.3 \sqrt[4]{\Delta\theta}$. For horizontal plates, facing upward, $h = 0.38 \sqrt[4]{\Delta\theta}$, and facing downward, $h = 0.2 \sqrt[4]{\Delta\theta}$. For flow outside horizontal pipes and long vertical pipes $h = 0.27 \sqrt[4]{\Delta\theta/D}$, where $D$ is the outside diameter of the pipe in inches.

The literature abounds with formulas for $h$ in the case of flow of the fluid, such as water or air, past the given surface. As is to be expected, the value of $h$ increases as the speed of the fluid relative to the contacted medium increases.

When a fluid flows past a solid surface, the fluid is normally at rest at the surface (if it wets the surface), and attains the speed of the fluid mass at some distance from the surface. There is thus a film or boundary layer through which the heat must be transferred.

**8.5. Heat transfer to a wall by convection.** Let us consider the example of Section 8.3 where there is heat transfer by convection from a fluid medium to each side of the panel.

The differential equation (8.5) still applies. Boundary conditions (8.6) apply, where now $\theta_1$ and $\theta_2$ are instantaneous panel surface temperatures. Let the fluid mass have the temperature $\theta_a$ on the $x = 0$ side of the panel and the temperature $\theta_b$ on the other side. Since the boundary layer is often quite thin such as a fraction of an inch, and with small heat capacity, we shall neglect storage of heat in this layer.

Consider an element of volume as in Figure 145 where now the face with the small $x$ coordinate is at $x = 0$. The rate of heat transfer due to convection to the face $x = 0$ of the element is $(\theta_a - \theta_1)h_1\Delta y\Delta z$ for a film coefficient $h_1$ of heat transfer. The rate of heat transfer due to conduction from this face is $-k\Delta y\Delta z(\partial\theta/\partial x)$. Since these must be equal we have

$$\frac{\partial\theta}{\partial x} = \frac{h_1}{k}(\theta_1 - \theta_a). \tag{8.10}$$

Similarly, for the surface $x = D$ we have

$$\frac{\partial\theta}{\partial x} = \frac{h_2}{k}(\theta_b - \theta_2) \tag{8.11}$$

for a coefficient $h_2$ of heat transfer. Equations (8.10) and (8.11) must be adjoined to the equations of Section 8.3.

**8.6. Radiation.** Heat energy can be transferred by electromagnetic waves, satisfying Maxwell's equations. The source of radiation, that is the radiator, emits electromagnetic waves of various wave lengths depending on the atomic structure of the radiator, as explained by quantum mechanics. Transfer of heat by radiation has the interesting property that this transfer can occur without heating objects between the sender and receiver of the radiation. Thermal radiation has a continuous spectrum for a black body, which will be defined later. Below 500 degrees centigrade the radiation is mostly infrared. We shall assume that the laws of geometric optics apply, so that there is no *diffraction* or *scattering*.

When radiant energy falls on a given body, the energy may divide in three ways. Part of it may be absorbed by the body. The fraction of the energy absorbed is called the *absorptivity*, and denoted by $\alpha_0$. Part of the energy may be reflected, the fraction reflected being the *reflectivity*, denoted by $\rho$. The remainder of the energy is transmitted through the body. The fraction transmitted is called the *transmissivity*, and is designated by $\tau$. Obviously, the sum of $\alpha_0$, $\rho$, and $\tau$ is 1.

In engineering we are often concerned with *opaque bodies* for which the transmissivity is practically zero. Except for gases, in what follows we treat only opaque bodies. If also the reflectivity is zero, the body is said to be a *black body*. The blackest surfaces in nature have about 1 per cent reflectivity. A practically perfect black body can be realized as follows. Consider a hollow evacuated body with the interior walls kept at a uniform temperature, and communicating with the outside through a circular hole whose dimensions are small compared with the dimensions of the cavity. Any radiation entering the hole is partly absorbed, and partly reflected a large number of times at the interior walls, only a small fraction eventually finding its way out of the hole. The circular plane region of the hole can be thought of as a good approximation to a black surface, for almost all radiation falling on this surface from the outside is absorbed in the cavity on the other side. This black body property is independent of the material of which the hollow body is made.

Experiments show that thermal radiation depends on the temperature of the emitting body, the surface area and roughness, as well as the atomic structure of the body. Experiments show also that the absorption, reflection, and transmission of thermal radiation depend on the same properties of the body. The rate at which radiant energy is emitted by a body does not depend on the surroundings. Experiments show that all bodies radiate energy at each temperature of the body.

**8.7. Kirchhoff's law.** Suppose that the walls of a hollow spherical cavity are kept at a temperature $T$, and that a black body $B$, such as a particle of coal, at the temperature $T$ is introduced into the cavity at the center, where $B$ is small compared with the cavity. To make the following argument rigorous we should take the limit as the diameter of the cavity increases beyond all bounds, so that because of symmetry we may suppose that at the center of the sphere the radiation becomes *isotropic*, that is, the radiation at (or near) the center is independent of the direction of the surface on which the radiation falls. Since the body $B$ and the walls of the cavity are at the same temperature $T$, they are in thermal equilibrium. It is assumed that the system is in steady state. Let $H$ be the rate at which radiant energy falls on the body $B$ per unit area, that is, $H$ is the *density* of the radiant energy rate falling on $B$. Let the rate at which the radiant energy is emitted by $B$ per unit area be denoted by $R_B$. Now the rate at which radiant energy is absorbed by $B$ equals the rate at which radiant energy is emitted by $B$, since the state is steady, whence

$$H = R_B. \tag{8.12}$$

Since this relation holds for each black body $B$ inserted in the cavity, the radiation $R_B$ depends only on the temperature. Isotropic radiation (the same for each point and direction) is termed *black body radiation.*

Suppose now that a nonblack body $C$ at the temperature $T$ is inserted into the above cavity. We let the surface density of the radiant energy emission rate from $C$ be designated by $R$. Since the system is in equilibrium we must have

$$\alpha_0 H = R \tag{8.13}$$

for the absorptivity $\alpha_0$ of $C$. By (8.12) and (8.13) we have

$$R = \alpha_0 R_B. \tag{8.14}$$

Equation (8.14) is a form of *Kirchhoff's law* of radiation. The ratio $R/R_B$ is called the *emissivity* of the body $C$. By definition the emissivity of the body $C$ is then the ratio of the thermal radiation rate per unit area of $C$ to this rate for a black body at the same temperature. Kirchhoff's law states that *the absorptivity and emissivity of a body are equal.* We let $\epsilon$ designate the emissivity.

Some emissivities follow. For iron in the range 150 degrees to 1,000 degrees centigrade, emissivity ranges correspondingly from 0.05 to 0.37. For copper at 100 degrees centigrade, emissivity is 0.018. For aluminum in the range 250 degrees to 600 degrees centigrade, emissivity varies correspondingly from 0.039 to 0.057. For gray rubber at 25 degrees centigrade we have $\epsilon = 0.86$. For wet ice at 0 degrees centigrade the emissivity is 0.97.

Emissivity thus depends on the temperature. For oxidized surfaces of metals, emissivity is greatly increased.

When the temperature of a body is the same as that of its surroundings the rate at which the body emits thermal radiation is equal to the rate at which it absorbs such radiation, so that there is no net change in the internal energy of the body due to thermal radiation. If, however, there is a temperature difference there will be a net gain or loss of thermal energy due to radiation. This net energy change due to the temperature difference is heat.

**8.8. Stefan-Boltzmann law.** Stefan verified experimentally that the rate $q_B$ at which thermal energy is radiated by a black body $B$ satisfies the relation

$$q_B = \sigma A T^4 \tag{8.15}$$

for the absolute temperature $T$ of the body, the surface area $A$, and a constant $\sigma$ of nature. The constant $\sigma$ is called *Stefan-Boltzmann's constant.* Boltzmann obtained (8.15) theoretically from thermodynamic considerations, whence (8.15) is known as the *Stefan-Boltzmann law.*

If we use British thermal units, hours, feet, and degrees Rankine, the Stefan-Boltzmann law becomes

$$q_B = 0.174A\left(\frac{T}{100}\right)^4,$$

where an approximate value is used for $\sigma$. The denominator 100 is convenient because radiation does not generally become relatively important until high temperatures like 1,000 degrees Rankine or more are attained.

In place of (8.15) we have

$$q_B = \epsilon\sigma A T^4 \tag{8.16}$$

for a body with the emissivity $\epsilon$.

To prove the Stefan-Boltzmann law it is assumed that the region of space, with which we are concerned, is filled with isotropic radiation.

Let $u$ be the energy density due to the radiation, and $p$ the pressure of radiation at a point of the region. By Theorem 5.1

$$p = \frac{u}{3}. \tag{8.17}$$

We assume that (8.17) is valid at each point in the space to be treated. Now black body radiation is described by the coordinates $p$, $V$, and $T$, where $V$ is the volume of the region of space being studied. We can therefore treat such a region of space as a thermodynamic system, and introduce its entropy. We suppose that the energy density $u$ is the same for all points in this space, whence the internal energy $U$ of the system is $uV$.

Consider an evacuated vertical cylinder with a black body at absolute temperature $T$ as base, a perfectly reflecting vertical wall, and a piston presenting a perfectly reflecting surface to the space $R$ bounded by the cylinder. The region $R$ is thus bounded above by the piston, below by the black body base, and laterally by the cylinder wall. Suppose that a weight is placed on the piston so that the pressure of radiation on the piston from the region $R$ is balanced by this weight. Let $V$, $U$, and $S$ denote the volume, total internal energy, and total entropy of the region $R$. By changing the weight on the piston slowly we can (at least theoretically) make the system undergo directly reversible processes.

From (8.17), $U = Vu$, and

$$dS = \frac{dU + pdV}{T}$$

we have

$$dS = \frac{udV + Vdu + \frac{u}{3}\,dV}{T} = \frac{\frac{4}{3}\,udV + V\frac{du}{dT}\,dT}{T}. \tag{8.18}$$

Since

$$dS = \frac{\partial S}{\partial T}\,dT + \frac{\partial S}{\partial V}\,dV,$$

it follows by (8.18) that

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{V}{T}\frac{du}{dT}, \qquad \left(\frac{\partial S}{\partial V}\right)_T = \frac{4}{3}\frac{u}{T}.$$

Now

$$\frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\cdot\frac{du}{dT}, \qquad \frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right)_T = \frac{4}{3}\,\frac{T\frac{du}{dT} - u}{T^2},$$

from which

$$\frac{du}{dT} = \frac{4u}{T}. \tag{8.19}$$

Integrating (8.18) gives

$$u = bT^4. \tag{8.20}$$

Suppose that we have a black body $B$ immersed in a homogeneous isotropic medium, where this medium occupies a portion $R$ of three-space, and the space occupied by this medium is filled with isotropic radiation. We assume that $B$ and the medium are in thermodynamic equilibrium. Let $R_1$ denote a portion of the surface of the body $B$. Suppose that $R_1$ is a plane region. Consider a plane region $R_2$ in the given medium such that the line segments which join points of $R_1$ to points of $R_2$ pass through the given medium only. The rate $Q$ at which radiant energy is transmitted from $R_2$ to $R_1$ is given by (5.27), where the symbols $A_1$, $A_2$, $\theta_1$, $\theta_2$, $r$, and $K$ are as defined in Section 5.4. The rate at which radiant energy goes from $R_1$ to $R_2$ equals the rate $Q$ at which radiant energy is transmitted from $R_2$ to $R_1$, for otherwise a one-sided transport of energy would be taking place. It follows that the total rate $q_B$ at which radiant energy is emitted by $B$ from $R_1$ is proportional to the brightness $K$ as well as the area $A_1$ of $R_1$. By (5.29) the brightness $K$ is proportional to the energy density $u$. In view of (8.20) we have $q_B$ proportional to $T^4$ and also proportional to $A_1$, so that a relation of the form (8.15) is valid.

If the region $R_1$ were not a portion of a plane, we should use a part of a plane tangent to $B$ instead of $R_1$, and apply limit methods.

We can extend the above considerations to the study of bands of frequencies.

We generally work with relatively opaque bodies for which $T$ in (8.16) can be taken as the surface temperature. Unless there is evidence to the contrary the radiation should be assumed to be isotropic so that this equation applies.

**8.9. Molten iron in an insulating container exposed to the air.** Suppose that we have a thermally insulating container filled with molten iron exposed to the air as in Figure 146. Since the temperature of the iron does not vary very much from point to point, we may suppose that it is uniform and equal to $T$ absolute. The air temperature is assumed uniform and given by $T_a$ absolute.

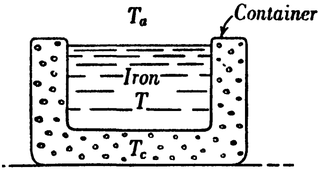

**Fig. 146. Molten iron in a container.**

We let $h$ be the coefficient of heat transfer between the iron surface and the air. We let $\epsilon$ be the emissivity of the iron, and $c_p$ the specific heat of the iron. It is supposed that the air is ordinary ambient air, such as air at a temperature of 70 degrees Fahrenheit. Since the air temperature is low, radiation from the air to the iron will be small compared to radiation from the iron to the air. The rate at which heat energy is radiated to the air is $\epsilon\sigma AT^4$ for the surface area $A$ of the iron, and Stefan-Boltzmann's constant $\sigma$. The rate at which heat is transferred from the iron to the air by convection is $hA(T - T_a)$. Let $t$ be time. The rate at which heat is lost from the iron is $c_p wV(dT/dt)$ for the specific weight $w$ and volume $V$ of the iron. It is assumed here that $c_p$ and $w$ are the same for different points in the iron, and that $dT/dt$ exists. Since all heat loss is through transfer to the air, we have the differential equation

$$wc_pV\frac{dT}{dt} + \epsilon\sigma AT^4 + hA(T - T_a) = 0. \tag{8.21}$$

In this equation we may suppose that $V$ and $A$ are constant, since the shrinkage of the iron during cooling is not very large, although by no means negligible in an accurate treatment. To solve (8.21) it is convenient to replace $T^4$ by $KT$ for an average value $K$ of $T^3$.

**8.10. Molten iron in a noninsulating container exposed to the air.** Let us suppose now that the container of Figure 146 is not composed of an insulating material.

We make the reasonable assumption that the temperature of the container is uniform, and given by $T_c$ absolute. We let $h_c$ be an average coefficient of heat transfer between the air and the container, and $h_i$ the coefficient of heat transfer between the iron and the container. We let $w_c$ be the specific weight of the container, $\epsilon_c$ the emissivity, and $c_{pc}$ the specific heat, these quantities being assumed to have the same values at all points of the container.

Through convection the rate of heat transferred from the iron to the container is $h_iA_c(T - T_c)$ where $A_c$ is the area of contact between the container and the iron. As a rough approximation we shall first neglect reflection of radiant energy. The net rate of heat transferred from the iron to the container through radiation is $\epsilon\sigma A_c(T^4 - \epsilon_c T_c^4)$. These losses must be added to the left of the equation (8.21) to give

$$wc_pV\frac{dT}{dt} + \epsilon\sigma AT^4 + hA(T - T_a) + h_iA_c(T - T_c) + \epsilon\sigma A_c(T^4 - \epsilon_cT_c^4) = 0 \qquad (8.22)$$

in place of (8.21).

Assuming that $dT_c/dt$ exists, the rate at which heat is stored in the container is $c_{pc}w_cV_c(dT_c/dt)$ for the volume $V_c$ of the material comprising the container. The net rate of heat transferred from the container by radiation to the molten iron is $\epsilon_c\sigma A_c(T_c^4 - \epsilon T^4)$, and this rate for convection from the iron is $h_iA_c(T - T_c)$. The container loses heat by radiation to the air at the rate $\epsilon_c\sigma A_o^*T_c^4$, where $A_c^*$ is the area of the container exposed to the air, and the radiation of the air to the material is again neglected. We are supposing that container material exposes the same type of surface to the air as to the iron. Otherwise we would have to use a different value of $\epsilon_c$ for the two surfaces. We thus have the following differential equation for the container:

$$c_{pc}w_cV_c\frac{dT_c}{dt} = \epsilon_c\sigma A_c(\epsilon T^4 - T_c^4) + h_iA_c(T - T_c) - \epsilon_c\sigma A_c^*T_c^4 - h_cA_c^*(T_c - T_a). \qquad (8.23)$$

On the left of (8.23) we have the rate at which heat is stored in the container. On the right we have the rate at which heat is received, minus the rate at which heat is lost.

The above approximations to the radiation rates may be too rough, in which event the powers $T^4$ and $T_c^4$ should be multiplied by constants depending on $\epsilon$ and $\epsilon_c$.

Equations (8.22) and (8.23) are the equations of the system in the variables $T$ and $T_c$. To solve these equations it is necessary to adjoin initial conditions. Such conditions follow:

$$t = 0, \quad T = T_0, \quad T_c = T_{c0}.$$

In deriving equations (8.22) and (8.23) we did not have to take account of heat conduction through the iron and container material because we assumed that the iron and container were each at uniform temperature. Had we not done this we would have obtained partial differential equations, whose solution would have been extremely difficult.

Equations (8.22) and (8.23) are undesirable in that they are nonlinear. We can linearize them however, by replacing $T^4$ by $kT$, where $k$ is an average value of $T^3$ for values of $T$ with which the analyst is concerned, and making a similar substitution for $T_c^4$.

The values of the constants, such as $h$ and $h_c$, can be secured from tables or by experiment.

### 8.11. Heat transfer to gas turbine blades by radiation, convection, and conduction.

Let us consider a gas turbine blade subjected to the temperature of a hot gas traveling past the blade at high speed (see Figure 147 for an aviation gas turbine). The blade is initially at a uniform temperature $T_{b0}$.

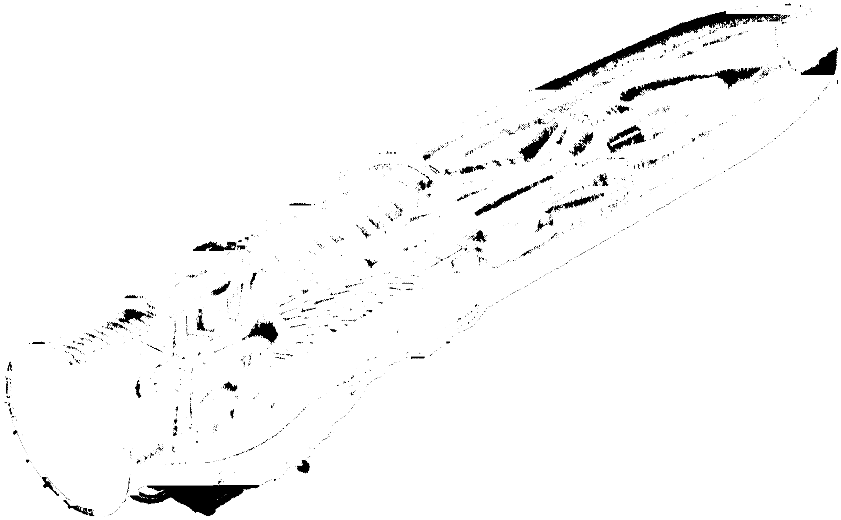

**Fig. 147. An aviation gas turbine. Courtesy of the Westinghouse Manufacturing Company.**

We let $T$ designate the absolute temperature at a point of the blade, and $T_g$ the absolute temperature of the gas.

A gas turbine blade is an airfoil as in Figure 148 where we have pictured a cross section of the blade. Not only does the cross section of the blade vary as one travels along its axis, but the blade is twisted about its axis. This twist is designed so that the air flow past the blade will have the correct absolute velocity. Since the air flow is reasonably the same past each cross section of the blade perpendicular to the axis of the blade, we may straighten the blade out. We also replace the cross section of the blade by a rectangle of about the same area, and width equal approximately to the average thickness of the airfoil. We thus obtain an *equivalent* parallelepiped as in Figure 149. We shall treat our problem as one-dimensional, so that $T$ is a function of a distance coordinate $z$ and time $t$ only.

We introduce a $z$-axis so that $z$ is in the direction of the length of the blade. The thickness of the blade is denoted by $D$, the width by $B$, and the length by $L$. We choose the origin so that $z = 0$ is at the face of the blade where the blade meets the hub of the turbine wheel. It follows that $z = L$ is at the tip of the blade.

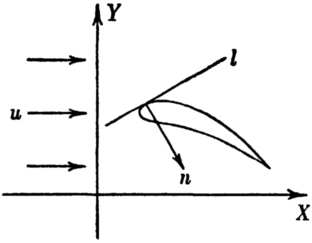

Fig. 148. Gas turbine blade.

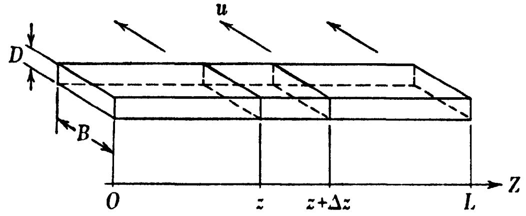

Fig. 149. Straightened blade.

For convenience the blade is shown in Figure 149 to be in a horizontal position. The speed of the air relative to the blade is denoted by $u$ in this figure. The velocity of the air is horizontal, and will vary with the distance from the root where $z = 0$, so that the coefficient of heat transfer between the air and the blade will vary accordingly. This coefficient will also vary from the leading edge to the trailing edge. We shall neglect this variation and use an average value of $h$ instead. We neglect the heat transfer across the edges, as well as the tip. The thermal conductivity $k$ of the blade is assumed to be uniform. The emissivity of the blade surface is denoted by $\epsilon_b$. The gas also has an emissivity $\epsilon_g$ which can be found from tables, at least approximately. The specific weight $w$ and specific heat $c_p$ are supposed to be uniform throughout the blade.

We consider the blade element between $z$ and $(z + \Delta z)$. We assume that the derivatives $\partial^2T/\partial z^2$ and $\partial T/\partial t$ exist and are continuous.

With $\sigma$ as the Stefan-Boltzmann constant, the rate at which heat is radiated from the blade element is $2\epsilon_b\sigma B(T_1^4 - \epsilon_g T_g^4)\Delta z$ for $T = T_1$ evaluated at some properly chosen point of the element, where $T_1$ is between the minimum and maximum values of $T$ for the points of the element. In this statement it is assumed that the blade emits more radiation than it receives. This is not necessary for the mathematical argument, because the plus or minus sign for the heat rate takes care of whether or not heat is lost or gained.

The rate at which the blade element gains heat by convection is $2hB(T_g - T_2)\Delta z$ for $T = T_2$ evaluated at some properly chosen point of the element. The rate at which heat is stored in the element is $wc_pBD(\partial T/\partial t)_a\Delta z$ for $\partial T/\partial t$ evaluated at some properly chosen section $z = a$ of the element.

The rate at which heat flows by conduction across the face $(z + \Delta z)$ of the element into the element is

$$kBD\left[\frac{\partial T}{\partial z}\right]_{z+\Delta z}.$$

The rate at which heat is conducted across the face $z$ of the blade element, toward the root, is

$$kBD\left[\frac{\partial T}{\partial z}\right]_z.$$

We assume that $T$ is such a function of $z$ that we can write

$$\left[\frac{\partial T}{\partial z}\right]_{z+\Delta z} = \left[\frac{\partial T}{\partial z}\right]_z + \left[\frac{\partial^2 T}{\partial z^2}\right]_z \Delta z + \eta, \tag{8.24}$$

where $\eta$ can be neglected in the limits to follow. The net rate of flow by conduction in the $z$-direction into the element is thus approximately

$$kBD\left[\frac{\partial^2 T}{\partial z^2}\right]_z \Delta z.$$

The rate of heat flow into the element due to radiation, convection and conduction is equal to the rate at which heat is stored in the element, plus the rate at which heat is radiated from the blade element. Thus

$$\begin{aligned} 2hB(T_g - T_2)\Delta z + kBD\left[\frac{\partial^2 T}{\partial z^2}\right]_z \Delta z &= wc_pBD\left[\frac{\partial T}{\partial t}\right]_a \Delta z \\ &\quad + 2\epsilon_b\sigma B(T_1^4 - \epsilon_g T_g^4)\Delta z. \end{aligned} \tag{8.25}$$

Dividing by $B\Delta z$, taking the limit as $\Delta z \to 0$, and rearranging terms, the following equation is obtained

$$wc_pD\frac{\partial T}{\partial t} + 2\sigma\epsilon_b T^4 + 2hT = 2hT_g + kD\frac{\partial^2 T}{\partial z^2} + 2\sigma\epsilon_b\epsilon_g T_g^4. \tag{8.26}$$

The hub of the turbine wheel is quite large compared with the blade, so that the blade temperatures will vary faster than the hub temperature. Unless we consider long periods of time, we may suppose that the hub temperature $T_h$ is constant. If we wish to be more accurate we can compute the temperature $T_h$ as a function of time $t$ from other considerations. In any case we have

$$z = 0, \quad T = T_h. \tag{8.27}$$

The temperature of the blade at the root is not actually $T_h$ because the blade is wedged into position so that heat must transfer across this junction. However, unless great accuracy is required, we can neglect the temperature differential across the junction. There is, of course, a transition area where the temperature varies from the temperature in the hub near the root of the blade to the temperature along the blade. Further, even though the hub temperature were mostly uniform, the hub material near a blade is bound to be of higher temperature than near the center of the hub.

The heat transfer at the tip was neglected in the treatment above. We might consider this tip to be insulated. Since the rate of heat conduction at the tip is proportional to $\partial T/\partial z$, the relation

$$z = L, \qquad \frac{\partial T}{\partial z} = 0 \tag{8.28}$$

applies.

Relations (8.26), (8.27), (8.28), as well as initial and final conditions such as $t = 0$, $T = T_{b0}$, describe the thermal conditions in the blade. More generally, we might have the initial condition $t = 0$, $T = T_0(z)$ for a given function $T_0(z)$.

An actual gas turbine blade is pictured in Figure 150.

### 8.12. Heat transfer to gas turbine blades taking into account the shape of cross sections of the blade.

We shall describe a still more accurate method of analyzing thermal conditions in a gas turbine blade. We now treat an actual cross section of the blade perpendicular to the blade axis. There is some variation in this cross section. We assume, however, that the cross section is constant and identical with some mean section of the actual blade. There is relatively little heat transfer at the blade tip, so that we may neglect this. In the mathematical treatment, for mathematical convenience only, we may consider that the blade is untwisted. This does not change the results of the analysis as they apply to the actual blade.

We introduce $x$- and $y$-axes in the plane of a cross section of the blade as in Figure 148. The $z$-axis, not shown, is perpendicular to the plane of the figure. If $L$ is the length of the blade, we take the root to be at $z = 0$, and the tip at $z = L$.

We derived equation (8.5) by assuming flow only through two parallel faces of the elementary parallelepiped. *In the case of flow across all faces this equation generalizes directly to*

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right], \tag{8.29}$$

where $\alpha$ is the thermal diffusivity of the blade. To complete the system of equations we must adjoin to the system initial, final, and boundary conditions, illustrated, in part, in the following.

In Figure 148 we have drawn a tangent line $l$ to the bounding curve of the cross section, and a normal ray $n$ to this tangent line toward the blade interior. The constants of the previous section may vary from point to point. This is particularly true of the coefficient of heat transfer $h$.

Fig. 150. A gas turbine blade. Courtesy of Fredric Flader Inc.

We consider a region $R$ on the surface of the blade, where this region has an area $A$. The rate of heat transfer to this region $R$ by convection is $hA(T_g - T)$ for a properly chosen value of $h(T_g - T)$ between the maximum and minimum values of this quantity for $R$, it being assumed that this function is continuous for $R$. The rate of heat loss from the region by radiation is $\epsilon_b\sigma A(T^4 - \epsilon_g T_g^4)$ for a properly chosen value of the continuous function $\epsilon_b(T^4 - \epsilon_g T_g^4)$ between the maximum and minimum values of this function taken on at the points of the region $R$. The rate at which heat is conducted away from the surface to the blade interior is $-kA(dT/dn)$ for the normal derivative $dT/dn$ of $T$ in the direction of the normal ray, and a properly chosen value of $k(dT/dn)$ between the maximum and minimum values of this function for $R$. It is assumed here that $dT/dn$ exists and is continuous.

We can express $dT/dn$ in terms of $\partial T/\partial x$, $\partial T/\partial y$, and the angle of inclination of $n$, which angle can be expressed in terms of $dy/dx$.

The rate of heat conducted away from the surface is equal to the rate at which heat is transferred to the surface by convection, minus the net rate at which heat is lost from the surface by radiation. We therefore have

$$k\frac{dT}{dn} + h(T_g - T) + \epsilon_b\sigma(\epsilon_g T_g^4 - T^4) = 0, \tag{8.30}$$

where $A$ has been divided out. Taking the limit as the maximum of the lengths of the chords joining points on $R$ goes to zero, equation (8.30) yields an equation of the same form valid at each point on the surface of the blade.

The initial temperature distribution in the blade is also given. For the hub temperature $T_h$ we have

$$z = 0, \qquad T = T_h.$$

Neglecting the heat transfer at the blade tip, we can consider that this tip is insulated, so that we have

$$z = L, \qquad \frac{\partial T}{\partial z} = 0,$$

the rate of heat conduction at the tip being proportional to $\partial T/\partial z$.

A more accurate analysis can be made by using the actual geometric shape of the blade. In general this would not be a reasonable thing to do because of the complications of the equations.

**8.13. Heat transfer in a stirring rod.** Suppose that we have a rod inserted into a hot bath perpendicular to the surface of the liquid as in Figure 151. If the cross section of the rod does not vary too much, we may suppose that it is constant and equal to $A$ in area. If the rod is bent we can straighten it for the analysis.

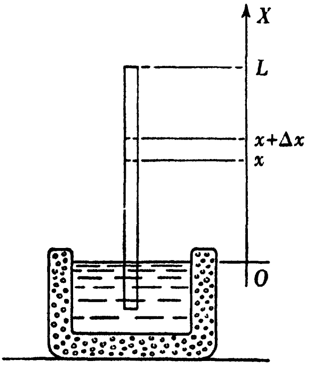

**Fig. 151.** Rod projecting from a liquid.

We set up an $x$-axis perpendicular to the surface of the liquid with the origin at the surface, and the positive sense directed upward. We let $L$ be the length of the rod exposed to the air. We suppose that the conductivity $k$ of the rod, the specific weight $w$, the emissivity $\epsilon$, and the specific heat $c_p$ are constant, and that we can use a uniform heat transfer coefficient $h$ for the heat transfer to the atmosphere to which the rod is assumed to be exposed. We let $T$ be the absolute temperature at a point of the rod. We let $P$ designate the perimeter of a cross section of the rod perpendicular to its axis.

We pass a plane through the rod perpendicular to the $x$-axis, and let $x$ be the coordinate of points in this section. We introduce a parallel section where $(x + \Delta x)$ is the coordinate of points in this section. We thus obtain an element $E$ of volume with lateral surface area $P\Delta x$ between the sections. We let $S_x$ denote the section associated with the coordinate $x$.

There is probably good contact between the liquid and the rod, so that we may suppose that the part of the rod immersed in the liquid is at the temperature $T_l$ of the liquid, it being supposed that $T_l$ is everywhere the same in the liquid, at least near the rod, and remains constant during the period of time covered by the analysis.

The temperature varies over a cross section of the rod. Since it is to be expected that most of the temperature variation is in the direction of the rod axis, we shall assume that we can consider the temperature over a cross section to be constant. Thus $T$ is a function of $x$ and time $t$ only. We assume that $\partial^2T/\partial x^2$ and $\partial T/\partial t$ exist and are continuous.

Consider the element of the rod between sections $S_x$ and $S_{x+\Delta x}$. The rate at which heat crosses the section $S_x$ minus the rate at which heat crosses the section $S_{x+\Delta x}$ can be written as

$$Ak\left\{\left(\frac{\partial T}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial T}{\partial x}\right)_x\right\}.$$

Except for a term which does not contribute to the final limit we can write this expression as

$$Ak\frac{\partial^2T}{\partial x^2}\Delta x.$$

The rate at which heat is transferred to the air by convection is $hP(T_1 - T_0)\Delta x$ for the temperature $T_0$ of the air and a properly chosen value $T_1$ of $T$ for points on the lateral surface $\Delta\Sigma$ of the element $E$. We neglect the radiation of the air to the rod. The radiation from the surface $\Delta\Sigma$ of the element $E$ of the rod to the air is at the rate $\epsilon\sigma PT_2^4\Delta x$ for a properly chosen value $T_2$ of $T$ for points on $\Delta\Sigma$, so that the rate of heat transfer to the air is

$$P[h(T_1 - T_0) + \epsilon\sigma T_2{}^4]\Delta x.$$

The rate at which heat energy is stored in the element $E$ is $wc_pA\,(\partial T/\partial t)\Delta x$

*for the derivative $\partial T/\partial t$ with respect to time $t$* evaluated at some properly chosen point of the element. Equating the rate of heat flow into the element $E$ to the sum of the rate of heat flow out of the element, and the rate at which heat is stored in the element, dividing both sides of the resulting equation by $\Delta x$, and taking the limit as $\Delta x$ goes to zero, we have

$$Ak\frac{\partial^2 T}{\partial x^2} = P[h(T - T_0) + \epsilon\sigma T^4] + wc_pA\frac{\partial T}{\partial t} \tag{8.31}$$

valid between $x = 0$ and $x = L$.

We have the boundary condition

$$x = 0, \qquad T = T_l.$$

The rate of heat flow to the end $x = L$ is equal to the rate at which heat is transferred to the air from this end of the rod, whence

$$x = L, \qquad k\frac{\partial T}{\partial x} + \epsilon\sigma T^4 + h(T - T_0) = 0. \tag{8.32}$$

To (8.31) and (8.32) we must adjoin initial and final conditions, such as the initial and final values of $T$.

**8.14. Heat conduction due to steam flow through a pipe.** Let us suppose that we have steam flowing through a right circular cylinder with inner radius $r$ and outer radius $R$. We let the $x$-axis be perpendicular to the axis of the cylinder with origin on this axis (see Figure 152). We shall use *combined coefficients of heat transfer for convection and radiation.* We let $L$ be the length of a section of the pipe under consideration, this section being bounded by planes perpendicular to the axis of the cylinder.

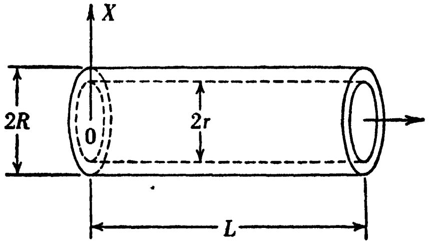

Fig. 152. Steam flow through a pipe.

We let $\theta_s$ be the temperature of the steam, and $\theta_0$ the temperature of the air, these temperatures being assumed to be uniform.

Let

$$r \leqq x < x + \Delta x \leqq R.$$

We consider a circular shell with inner radius $x$ and outer radius $(x + \Delta x)$, and axis identical with the axis of the pipe. We let $\theta$ be the temperature at a point of the pipe, and suppose that $\partial^2\theta/\partial x^2$ and $\partial\theta/\partial t$ exist and are continuous. The rate of heat flow across the inner surface of the shell is $-2\pi xkL(\partial\theta/\partial x)_x$, where $(\partial\theta/\partial x)_x$ designates the derivative $\partial\theta/\partial x$ evaluated at the inner surface, and $k$ is the conductivity. For the outer surface we have the corresponding value $-2\pi(x + \Delta x)kL(\partial\theta/\partial x)_{x+\Delta x}$. The rate at which heat energy is stored in the element is $wc_p\pi L[(x + \Delta x)^2 - x^2](\partial\theta/\partial t)_\xi$ for the specific weight $w$ and specific heat $c_p$ of the pipe material, assumed to be uniform, and the derivative $\partial\theta/\partial t$ evaluated at some coordinate $\xi$ between $x$ and $(x + \Delta x)$. We have the heat balance equation

$$2k\left[(x + \Delta x)\left(\frac{\partial\theta}{\partial x}\right)_{x+\Delta x} - x\left(\frac{\partial\theta}{\partial x}\right)_x\right] = wc_p(2x + \Delta x)\left(\frac{\partial\theta}{\partial t}\right)_\xi \Delta x.$$

Replacing $(\partial\theta/\partial x)_{x+\Delta x}$ by $(\partial\theta/\partial x)_x + (\partial^2\theta/\partial x^2)_x\Delta x + \eta$, dividing both sides of the above equation by $\Delta x$, and taking the limit as $\Delta x$ goes to zero, we have

$$k\left[x\frac{\partial^2\theta}{\partial x^2} + \frac{\partial\theta}{\partial x}\right] = wc_p x\frac{\partial\theta}{\partial t}. \tag{8.33}$$

Dividing both sides of (8.33) by $x$, with $x \neq 0$, we obtain a well-known form of an equation whose solution can be found in the literature.

The rate of heat energy transferred to the pipe by the steam is given by $2\pi rLh_s(\theta_s - \theta_1)$ for the combined radiation-convection coefficient of heat transfer $h_s$ from the steam to the wall and the temperature $\theta_1$ of the inner wall of the pipe. The rate at which heat energy is transferred to the interior of the pipe is $-2\pi rLk[\partial\theta/\partial x]_r$. Since these rates are equal we have

$$x = r, \qquad k\frac{\partial\theta}{\partial x} = h_s(\theta - \theta_s).$$

For heat transfer to the air we have the corresponding equation

$$x = R, \qquad k\frac{\partial\theta}{\partial x} = h_0(\theta_0 - \theta)$$

where $h_0$ is the combined radiation-convection coefficient of heat transfer.

We shall give a simplification of the above theory. Let us suppose that the temperature in the pipe wall is uniform, an assumption certainly valid if the pipe wall is not too thick. In this case the rate at which heat energy is stored is $\pi wc_pL(R^2 - r^2)\partial\theta/dt$. The rate at which heat energy enters the pipe through the inner surface is equal to the rate at which heat energy is stored, plus the rate at which heat energy is dissipated to the air. Thus

$$2rh_s(\theta_s - \theta) = wc_p(R^2 - r^2)\left(\frac{d\theta}{dt}\right) + 2Rh_0(\theta - \theta_0).$$

This differential equation is easily solved in conjunction with an initial condition.

**8.15. Principles in the analysis of heat.** We shall state some helpful principles for the rapid treatment of heat problems.

When feasible the given bodies should be simplified geometrically, and the problem reduced to a minimum number of geometric dimensions. The bodies should be assumed to be of uniform density, conductivity, specific heat, and emissivity, and to possess uniform coefficients of heat transfer at surfaces of contact with fluids, unless otherwise specified. The temperature in each body should be taken uniform unless the body is a large one with parts of its surface exposed to quite different temperatures.

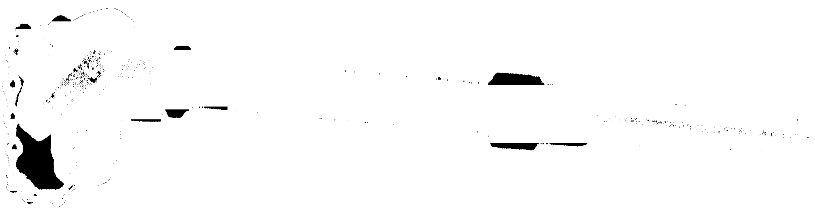

Fig. 153. A thermocouple unit for measuring temperature. Courtesy of the Leeds and Northrup Company.

For solids in contact with liquids these substances should be assumed, when conditions permit, to have the same temperature at the surface of contact.

If a substance is in the gaseous phase, it often can be treated as a perfect gas, and corrections made as indicated. If in a gaseous mixture one gas, such as air, predominates by weight, and one is studying thermodynamic transitions of the mixture from one equilibrium state to another, the mixture should be considered to be composed entirely of the dominating element.

In first analyses friction losses should be neglected, and corrected for later.

Where a substance passes rapidly through a region bounded by a physical body, the changes of state may be assumed to take place adiabatically. Where gases are not moving rapidly it may often be supposed that they are uniform as to pressure, volume, and temperature. Even where rapid changes take place, one should try to use average values of the variables, and employ thermodynamic theory. Where the mass rate of flow of a gas through a region is large, the deviations in the amount of the gas stored in the region should be neglected.

**8.16. Dimensional analysis applied to a free convection problem.** Suppose that we wish to find a formula for the coefficient of heat transfer by free convection between a vertical wall and a warm fluid. We take the primary quantities to be heat $H$, time $t$, length $L$, mass

$M$, and temperature $\theta$. The secondary quantities and their dimensions are given below.

| SECONDARY QUANTITY | $H$ | $t$ | $L$ | $M$ | $\theta$ |
|---|---|---|---|---|---|
| Heat transfer coefficient $h$ | 1 | −1 | −2 | 0 | −1 |
| Height $l$ of the wall | 0 | 0 | 1 | 0 | 0 |
| Temperature difference $\Delta\theta$ between the wall surface and the fluid | 0 | 0 | 0 | 0 | 1 |
| Fluid density $\rho$ | 0 | 0 | −3 | 1 | 0 |
| Viscosity $\mu$ of the fluid | 0 | −1 | −1 | 1 | 0 |
| Thermal coefficient $\beta$ of expansion of the fluid | 0 | 0 | 0 | 0 | −1 |
| Magnitude $g$ of the acceleration of gravity | 0 | −2 | 1 | 0 | 0 |
| Specific heat $c_p$ of the fluid when $w$ in (8.3) is replaced by $\rho$ | 1 | 0 | 0 | −1 | −1 |
| Thermal conductivity $k$ of the fluid | 1 | −1 | −1 | 0 | −1 |

We form the product

$$h^{a_1}\, l^{a_2}\, (\Delta\theta)^{a_3}\, \rho^{a_4}\, \mu^{a_5}\, \beta^{a_6}\, g^{a_7}\, c_p^{a_8}\, k^{a_9}. \tag{8.34}$$

By the Buckingham pi theorem we expect four independent dimensionless products. Imposing the condition that the term (8.34) be dimensionless we have the system of equations

$$\begin{aligned}
a_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + a_8 + a_9 &= 0,\\
-a_1 \qquad\qquad\qquad - a_5 \qquad\quad - 2a_7 \qquad\quad - a_9 &= 0,\\
-2a_1 + a_2 \qquad - 3a_4 - a_5 \qquad\quad + a_7 \qquad\quad - a_9 &= 0,\\
a_4 + a_5 \qquad\qquad\qquad - a_8 \qquad\quad &= 0,\\
-a_1 \qquad + a_3 \qquad\qquad\qquad - a_6 \qquad\quad - a_8 - a_9 &= 0.
\end{aligned} \tag{8.35}$$

We can choose four of the $a$'s arbitrarily. We make the following choices corresponding to the indicated $\pi$-terms.

| | $a_1$ | $a_7$ | $a_6$ | $a_8$ |
|---|---|---|---|---|
| $\pi_1$ | 1 | 0 | 0 | 0 |
| $\pi_2$ | 0 | 1 | 0 | 0 |
| $\pi_3$ | 0 | 0 | 1 | 0 |
| $\pi_4$ | 0 | 0 | 0 | 1 |

Solution of equations (8.35) for the remaining $a$'s gives

$$\pi_1 = \frac{hl}{k}, \qquad \pi_2 = \frac{l^3\rho^2 g}{\mu^2}, \qquad \pi_3 = \beta \cdot \Delta\theta, \qquad \pi_4 = \frac{\mu c_p}{k}.$$

Thus

$$h = \frac{k}{l} f\left(\frac{l^3\rho^2 g}{\mu^2}, \beta \cdot \Delta\theta, \frac{\mu c_p}{k}\right).$$

The quantity $\mu c_p/k$ is the famous *Prandtl number* $N_{Pr}$. When we associate a diameter $D$ with a system as a characteristic length we have the following well-known dimensionless groups for the theory of heat transfer:

| | |
|---|---|
| Reynolds number $R$ | $\dfrac{vD\rho}{\mu}$ |
| Grashof number $N_{Gr}$ | $\dfrac{D^3\rho^2\beta g \cdot \Delta\theta}{\mu^2}$ |
| Nusselt number $N_{Nu}$ | $\dfrac{hD}{k}$ |

The symbol $v$ in $R$ denotes speed.

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# Problems for Part III

The reader is expected to set the following physical systems in mathematical form, but is not supposed to solve the resulting equations. In these problems we are interested in heat transfer, except where thermodynamics is specifically mentioned.

1. A container of uniform thickness and material is spherical. It contains air, and is being uniformly heated.
2. The rod of Figure 151 has its top end insulated.
3. The bottom of a metal plate is being uniformly heated by hot gases; the top is exposed to the atmosphere.
4. A cylindrical rod, initially at a uniform temperature, is moved parallel to itself in still air with a constantly accelerating motion. The temperature of the air is uniform.
5. The rod of Problem 4 is suspended in still air and is at rest with respect to the air. Initially the temperature of the rod is higher than that of the air.
6. A jet plane encounters considerable skin friction at high speed. We are interested in the skin temperature at a fixed high speed.
7. A meteor enters our atmosphere at tremendous speed, and is being heated by air friction.
8. An athodyd (Figure 154) travels at very high speed, propelled by burning gases. Air rammed into the athodyd furnishes the oxygen for combustion.

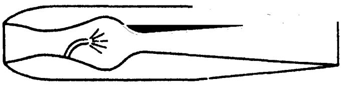


**Fig. 154. Athodyd.**

9. A flatiron is heated by currents through internal resistors.
10. An electric bulb is being heated from the electric current of the bulb.
11. A balloon is rising in the air while considerable heat is being lost to the atmosphere through the fabric.
12. A gas is condensing in a condenser composed of coils in the usual manner.
13. A brake shoe applied to an automobile wheel at constant pressure slides on the wheel and heats up.
14. An open container $B$ is filled with a liquid and rides on a liquid bath in an open container $A$. The surfaces of $A$ and $B$ are exposed to the atmosphere. The container $A$ is being heated by a fire underneath.
15. A cool room is surrounded by air at a uniform hotter temperature.

16. One end of a rod is immersed in a bath at one temperature, and the other end in a bath at another temperature. Between the ends the rod is exposed to the air. Initially the rod is at a uniform temperature.
17. A small hot circular plate radiates heat to a large hot circular plate at a different temperature, the plates being parallel and so placed that the line joining the centers is perpendicular to the plates.
18. The bulb of a glass thermometer is suddenly placed in a hot bath.
19. A glass thermometer is suddenly moved from inside a building to the outside.
20. A fire is heating a boiler containing water, and steam generated passes through a hollow tube to a nozzle from which it emerges.
21. A bowl of water with surface exposed to the air is heated by a submerged electric bulb.
22. The gas in an automobile cylinder is being cooled by a water jacket.
23. The cylinder of an aircraft engine is being cooled by an airstream. The cylinder has metal cooling fins.
24. One side of a one-room house is a window pane exposed to the sun. The room is becoming warmer due to the radiation from the sun.
25. A hot piece of iron is thrown into a large water bath.
26. Water, initially cool, flows through a long pipe exposed to warm air until it reaches a faucet. We are interested in the emergence temperature.
27. A hollow sphere containing a gas is suddenly immersed in a hot bath.
28. A fixed voltage is suddenly applied across the terminals of the heating element of an electric toaster. Analyze the heat generated.
29. A steam turbine is running. We are interested in the thermodynamic processes.
30. A reciprocating steam engine is in operation. We are interested in the thermodynamic processes.
31. A two-stroke diesel engine is running. We are interested in the thermodynamic processes.
32. A pile of powdered coal is burning.
33. Water in an open container is heated by a fire underneath, and boils.
34. The container in Problem 33 is a pressure cooker.
35. By dimensional analysis find a formula for the rate of heat transfer between a hot rigid body submerged in a flowing fluid and this fluid.
36. A long, straight, smooth pipe of uniform diameter is given. Consider the rate $dp/dl$ of pressure drop, due to friction, for isothermal turbulent flow of an incompressible fluid at a constant mass rate through the pipe. Here $p$ denotes pressure, and $l$ a length coordinate. Find a formula for the rate of pressure drop.
37. Heat is generated uniformly in a homogeneous right circular cylinder. Find the steady state temperature difference $\Delta\theta$ between the center and the outer surface.

# PART IV

# ELASTICITY

## CHAPTER

## 9

# Fundamental theory of elasticity

**9.1. Internal forces.** In Part IV we shall be concerned with transforming physical problems involving elastic bodies into mathematical form.

If a body $B$ is in *static* equilibrium, any subbody $B_s$ of $B$ is in equilibrium under the influence of the forces acting on it due to the rest of the body, and the external forces. In studying the behavior of $B$ in its interior due to applied forces, it is much more convenient for an understanding of the theory to treat what happens at points in $B$ than to treat subbodies of $B$, since, if we know the situation at each point of $B$ we know the mechanical state of $B$. On the other hand, for solution of actual engineering problems, study of subbodies often leads to quick results.

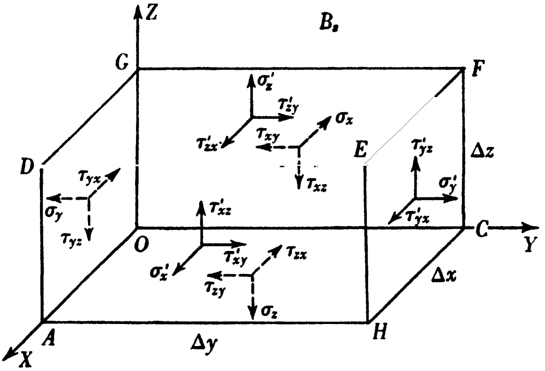


**Fig. 155. Stresses on an element B$_s$.**

We select an arbitrary point $O$ in the body $B$, and with this as origin set up a Cartesian coordinate system as in Figure 155. In practice a point in $B$ may be thought of as the *position* occupied by an individual molecule in $B$ so that we can follow movements of points in $B$, even though a molecule is not a particle and most of $B$ is empty space between atomic particles. We make this remark because we are going to study bodies which change shape and it is necessary to trace movements of points in these bodies. It is convenient also to think of the body as a continuous distribution of matter.

We shall include inertial forces in our treatment, so that we are considering both static and dynamic equilibrium. We assume that $O$ is an interior point in $B$, or is on the surface of $B$ in such a way that the rectangular parallelepiped $B_e$ of Figure 155 lies in $B$. If $O$ is on the surface of $B$ we may use a modified element, one face of which is part of the surface of the body $B$.

We suppose that the magnitude of the force $F_s$ acting on a subregion of a plane region $R$ (which subregion might be the region $R$ itself) is proportional to the area of the subregion, and that this force is always in the same direction. Let $A$ be the area of $R$. If $F$ denotes the force acting on $R$, the vector $F/A$ is the *stress* associated with each point of $R$. If the properties assumed for $F_s$ do not hold, or $R$ is not plane, we can by limit methods associate a stress with each point of $R$.

The resultant force applied to the face $AOGD$ of $B_e$ by the material adjacent to (on the left of) this face can be resolved into its components parallel to the coordinate axes. Dividing the functions representing these components by the area $\Delta x \Delta z$ of the face $AOGD$ we obtain $\tau_{yx}$, $\sigma_y$, and $\tau_{yz}$ as shown; similarly, for the other faces. The quantities $\tau_{yx}$ and $\tau_{yz}$ in the directions of the $x$- and $z$-axes represent average *shear stresses* for $AOGD$. The quantity $\sigma_y$ represents a *compression* or *tension* (tensile) *stress* depending on whether or not $\sigma_y$ is directed inward or outward from $B_e$. The positive directions are taken as shown. We shall use primes on the symbols for the stresses on the planes of the element $B_e$ that are not coordinate planes.

Besides the *surface forces* represented by $\sigma_y \Delta x \Delta z$, etc., there may be *body forces* acting on $B_e$, such as gravitation and magnetism. We let $X\Delta V$, $Y\Delta V$, and $Z\Delta V$ represent the $x$-, $y$-, and $z$-components of the resultant of the body forces acting on $B_e$ where $\Delta V$ is the volume of $B_e$.

We let $\rho$ denote the density at a point of $B_e$. We suppose that each point in $B_e$ has an *acceleration* that can be determined, the $x$-component being represented by $x''$. We assume that $\rho x''$ varies continuously over the points of $B_e$. For a properly chosen value $\bar{\rho}\bar{x}''$ of $\rho x''$ between the minimum and maximum values of $\rho x''$ we can represent the inertial force on $B_e$ in the $x$-direction by $\bar{\rho}\bar{x}''\Delta x \Delta y \Delta z$.

From Newton's second law we have

$$(\tau'_{yx} - \tau_{yx})\Delta x \Delta z + (\tau'_{zx} - \tau_{zx})\Delta y \Delta x + (\sigma'_x - \sigma_x)\Delta y \Delta z + X\Delta x \Delta y \Delta z = \bar{x}''\bar{\rho}\Delta x \Delta y \Delta z. \tag{9.1}$$

Dividing by $\Delta x \Delta z$, taking the limit as $\Delta y$ goes to zero, assuming that the limits of the terms exist, (9.1) yields

$$\tau_{yx} = \tau'_{yx}. \tag{9.2}$$

Letting the maximum of $|\Delta x|$ and $|\Delta z|$ now go to zero we obtain (9.2) for the stresses $\tau_{yx}$, $\tau'_{yx}$ at the point $O$. Similarly, dividing both sides of (9.1) by $\Delta y \Delta x$ and letting $\Delta z$ go to zero, we have $\tau_{zx} = \tau'_{zx}$, etc. For this reason, in the literature on the subject, the primes are not placed on the symbols representing stresses, although it is incorrect to start with the assumption that the unprimed functions representing average shear stresses equal the corresponding primed functions numerically. In general, for $\Delta x \neq 0$, $\Delta y \neq 0$, and $\Delta z \neq 0$ the stresses represented by primed symbols are different from the stresses represented by unprimed. *It is in the limit that they are equal.* Assuming continuity of the stresses, the limit process thus yields the functions

$$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yx},\ \tau_{xz},\ \tau_{zx},\ \tau_{yz},\ \tau_{zy} \tag{9.3}$$

representing stresses at the point $O$. The functions (9.3) can be conveniently arranged in the *matrix*

$$\begin{Vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{Vmatrix} \tag{9.4}$$

defining the *stress tensor*, and will be called *stress components*. That the entries in (9.4) are not independent will be shown later.

As $\Delta x$, $\Delta y$, and $\Delta z$ go to zero simultaneously, the limit of $X$ (if it exists) defines the $x$-component of the *body force per unit volume at the point O.*

As far as the forces are concerned, *the state (mechanical) of the body B is determined* by the stresses, body forces, and density at each point, it being possible to obtain the forces acting on any subbody of $B$ by the usual integration methods.

**9.2. Elastic bodies.** In the analysis of physical systems it is convenient to idealize the bodies involved by describing them mathematically in terms of their dominant properties. We know what the dominant properties are from experience, or measurements made on the bodies being studied. To be able to carry through mathematical computations the properties describing a body should, of course, be as simple as possible. In Parts I and V we treat two extremes, namely the *rigid body* where the distance between any two points in the body does not change, and the *perfect fluid* characterized by the property that it can support no stresses other than hydrostatic pressure, that is, the pressure $p$ of Part V. In the present chapter we shall be concerned with elastic bodies whose properties lie between those of the fluid and those of the rigid body. We shall also make some remarks on plastic materials. Whether or not a given body is to be treated as a fluid, elastic, plastic, or rigid body naturally depends on the nature of the problem being studied. Thus if a piece of iron in a machine is subjected to stresses of 1 pound per inch, the resulting deformations are generally negligible, on the order of $3 \cdot 10^{-8}$ inch, and the piece can therefore be considered rigid, whereas under stresses of 10,000 pounds per inch, the deformations may be large enough to spoil the fit between the given piece and other members of a machine, so that in this case the elastic properties must be considered.

An *elastic body* is one which, when deformed from an initial state $A$ to a state $B$ under the influence of forces, resumes its initial state $A$ when these forces are removed. By *state* we refer to the geometric shape and mechanical properties, namely, the density and stresses at each point. There are no perfectly elastic bodies. If sufficiently great forces are applied, the elastic limit may be exceeded, and the body will not return to its original state. Even though the forces are small, fine geometrical measurements will show an altered shape when these forces are removed. Whether or not a body is to be considered elastic in a given problem depends on the materials of which the body is made, the magnitude of the forces or stresses involved, the geometric magnitudes which can be neglected, as well as other factors. Experience and experiment are very important in this matter.

We are often concerned with elastic bodies that are considered to be initially made of the same material throughout and of uniform density, and which for small deformations do not deviate much from these properties.

As we have done above we shall distinguish between surface and body forces that act on a given body $B$. A body force is known by its intensity at each point. Thus if $X$, $Y$, and $Z$ represent the $x$-, $y$-, and $z$-components of the body force intensity at a point of $B$, the $x$-, $y$-, and $z$-components of the body force acting on any portion $B_1$ of $B$ are represented by

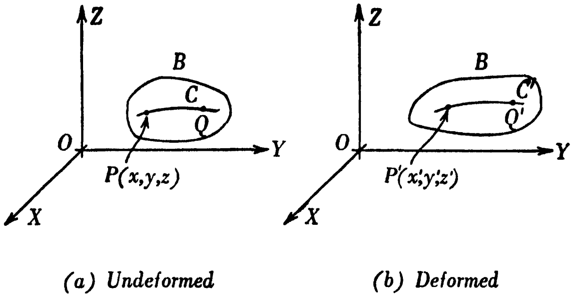
$$\iiint_{B_1} X dV, \quad \iiint_{B_1} Y dV, \quad \iiint_{B_1} Z dV$$

integrated over the portion $B_1$, the element of volume being denoted by $dV$. Surface forces will also be given by their intensities, that is, stresses, so that if $X$, $Y$, and $Z$ represent the $x$-, $y$-, and $z$-components of a surface force intensity, the surface force acting on a portion $S_1$ of the surface $S$ of $B$ has the $x$-, $y$-, and $z$-components represented by

$$\iint_{S_1} X d\Sigma, \quad \iint_{S_1} Y d\Sigma, \quad \iint_{S_1} Z d\Sigma$$

respectively, where $d\Sigma$ denotes element of surface area.

**9.3. Strains.** We shall consider a body $B$ given undeformed in Figure 156 (a), and deformed as indicated by Figure 156 (b). In each case the body $B$ is shown with reference to an $xyz$-Cartesian coordinate system. We shall suppose, for the purpose of discussion, that we know exactly how the points in the body $B$ shown in the first figure correspond to the points of this body in the second. It is understood that the deformation is a continuous one, without cutting or tearing, so that the curve $C$ drawn through the point $P(x, y, z)$ of the first figure deforms into a curve $C'$ in the second figure, where $P'(x', y', z')$ is the new position of $P$. We assume that these curves are rectifiable so that the points on $C$ can be designated in terms of an arc length coordinate $s$ measured from some point on $C$, and that there is a similar coordinate $s'$ for $C'$. In what follows the variables will be considered to be functions of $s$.



(a) Undeformed (b) Deformed

Fig. 156. Body *B*.

We introduce the *displacements* $u$, $v$, $w$ by means of the relations

$$x' = x + u, \quad y' = y + v, \quad z' = z + w,$$

whence these displacements are the differences between corresponding coordinates of $P$ and $P'$. The deformation can be recognized through a study of $u$, $v$, and $w$.

A section of an unstressed elastic homogeneous rod is of length $a$, and is stretched to a length $b$ by stresses at the ends of the rod. For small deformations experiment shows that $b$ increases with $a$ in direct proportion so that $\epsilon$ given by

$$\epsilon = \frac{b - a}{a}$$

is constant; that is, the relative elongation is constant. We therefore call $\epsilon$ the *strain* imposed on the rod by the given stresses.

We let $Q$ be a point on $C$, and $Q'$ the corresponding point on $C'$. We recognize that the body $B$ in Figure 156(b) is strained, relative to its state in Figure 156(a), by the property that for a curve $C'$ and points $P'$ and $Q'$ on $C'$ the distance $\Delta s'$ from $P'$ to $Q'$ is different from the distance $\Delta s$ from $P$ to $Q$ on $C$. The quantity

$$\frac{\Delta s' - \Delta s}{\Delta s} \tag{9.5}$$

is a measure of the strain along $C'$ between $P'$ and $Q'$. Taking the limit as $\Delta s$ goes to zero (assuming that the limit of $\Delta s'/\Delta s$ exists) the ratio (9.5) yields the *strain e at the point P'* on $C'$, where

$$e = \frac{ds' - ds}{ds}. \tag{9.6}$$

The formula (9.6) defines a strain $e$ for each point of $B$ and direction through this point.

We shall obtain a formula for $e$ in terms of the displacements $u$, $v$, and $w$. We assume that the displacements are due to deformation only. We thus exclude the case where the body is translated and rotated as a rigid

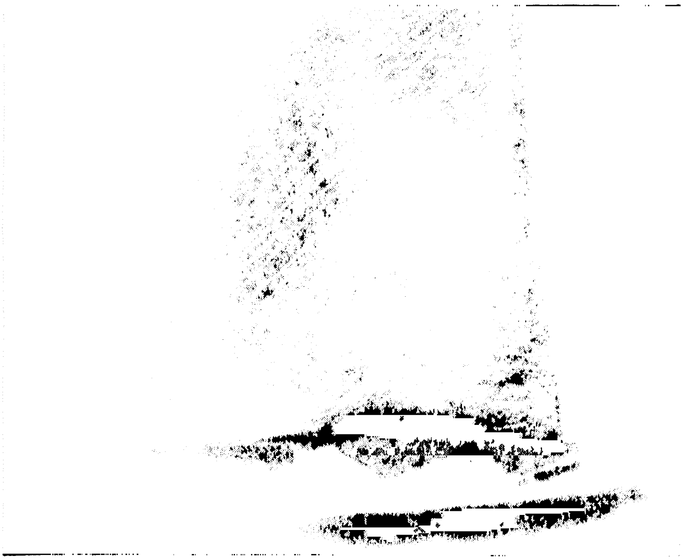


**Fig. 157. Stress crack pattern in the coating of an automobile wrench magnified 2.5 diameters. Specimen supplied by Professor W. M. Murray of the Massachusetts Institute of Technology was prepared by Mr. G. H. Eisenhardt. Photograph by Mr. Leon Keinigsberg.**

body. We suppose that all derivatives that follow exist. The direction cosines $l$, $m$, and $n$ of the tangent to $C$ at $P$ are given by

$$l = \frac{dx}{ds}, \quad m = \frac{dy}{ds}, \quad n = \frac{dz}{ds},$$

and the direction cosines $l'$, $m'$, and $n'$ of the tangent line to $C'$ at $P'$ are given by

$$l' = \frac{d(x+u)}{ds'}, \quad m' = \frac{d(y+v)}{ds'}, \quad n' = \frac{d(z+w)}{ds'}.$$

From

$$\frac{d(x+u)}{ds} = \frac{dx}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} + \frac{\partial u}{\partial z}\frac{dz}{ds}$$

we have

$$l' = \frac{ds}{ds'}\left[\left(1 + \frac{\partial u}{\partial x}\right) l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n\right]. \tag{9.7}$$

We obtain similar relations for $m'$ and $n'$. Substituting the formulas of type (9.7) in

$$(l')^2 + (m')^2 + (n')^2 = 1$$

we obtain

$$\left(\frac{ds'}{ds}\right)^2 = (1 + 2\epsilon_x)l^2 + (1 + 2\epsilon_y)m^2 + (1 + 2\epsilon_z)n^2 + 2\gamma_{xy}lm + 2\gamma_{zx}ln + 2\gamma_{yz}mn, \tag{9.8}$$

where

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\},\\
\epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\},\\
\epsilon_z &= \frac{\partial w}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\},\\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},\\
\gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z},\\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}.
\end{aligned} \tag{9.9}$$

From (9.8) we can compute the strain $e$ of (9.6) for each point $P$ of $B$ and direction through this point. We note that $e$ can be written as

$$e = \frac{ds'}{ds} - 1.$$

Experiment shows that in the ordinary treatment of elastic bodies the products of the derivatives of $u$, $v$, and $w$ with respect to $x$, $y$, and $z$ are small compared to these derivatives, so that (9.9) can be simplified to the approximate relations

$$\begin{gathered}
\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_z = \frac{\partial w}{\partial z},\\
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}.
\end{gathered} \tag{9.10}$$

The quantities $\epsilon_x$, $\epsilon_y$, $\epsilon_z$, $\gamma_{xy}$, $\gamma_{zx}$, and $\gamma_{yz}$ are called the *components of strain*. These components, as given in (9.10), yield the *strain tensor*

$$\left\| \begin{matrix} \epsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \epsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \epsilon_z \end{matrix} \right\| \tag{9.11}$$

analogous to the stress tensor (9.4); here $\gamma_{xz} = \gamma_{zx}$. By convention we have also $\gamma_{zy} = \gamma_{yz}$, $\gamma_{yx} = \gamma_{xy}$.

In Section 9.1 the stresses were referred to a point $(x, y, z)$ of the deformed material, whereas the coordinates $x$, $y$, and $z$ of (9.10) designate a point in the undeformed body. Because we shall later give relations between stresses and strains it is convenient to refer both stresses and strains to coordinates $x, y, z$ of the deformed body existing under the given stress distribution and body forces. We note that the coordinates $(x + u)$, $(y + v)$, and $(z + w)$ of the deformed body $B$ are functions of $x$, $y$, and $z$. Since

$$\frac{\partial(x + u)}{\partial x} = 1 + \frac{\partial u}{\partial x},$$

we have, when $u$ is taken as a function of $(x + u)$, $(y + v)$, and $(z + w)$,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial(x + u)}\left(1 + \frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial(y + v)}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial(z + w)}\frac{\partial w}{\partial x}. \qquad (9.12)$$

We assume that the derivatives are small enough so that, for example, $\left(1 + \frac{\partial u}{\partial x}\right)$ can be replaced by 1 in the first term on the right of equation (9.12), and products of derivatives are negligible compared to derivatives. In this case we can write $\frac{\partial u}{\partial(x + u)}$ in place of $\frac{\partial u}{\partial x}$. Such an argument shows that we may take the coordinates $x$, $y$, $z$ in (9.10) to designate a point in the deformed body $B$ of Figure 156(b).

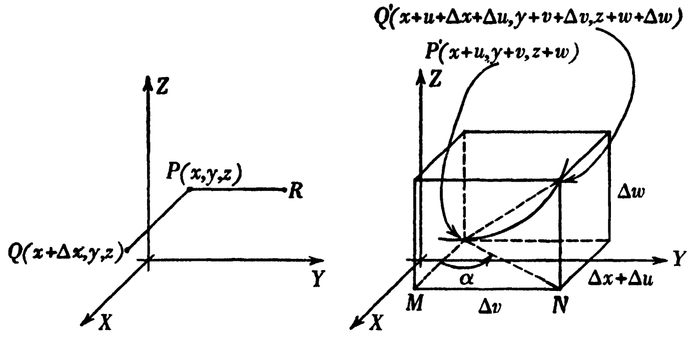
When we relate stresses to strains in what follows, the coordinates $x$, $y$, $z$ will refer to a point in the deformed body.

The six components (9.10) of strain are dependent on the three functions $u$, $v$, and $w$ only, and we would therefore expect the strains to be subject to various conditions. It is easily verified by direct differentiation of the strains that the following *compatibility equations* hold:

$$\begin{aligned}
&\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y},\\
&\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z},\\
&\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x},\\
&2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right),\\
&2\frac{\partial^2 \epsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right),\\
&2\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right),
\end{aligned} \qquad (9.13)$$

assuming that the order of differentiation is immaterial.

**9.4. Geometric interpretation of strains.** We shall give a simple interpretation of the strain components in (9.10). We consider two points $P$ and $Q$ of the undeformed body $B$ on a line parallel to the $x$-axis as shown in Figure 158(a), these points being designated by $(x, y, z)$ and $(x + \Delta x, y, z)$ respectively. The corresponding points of the deformed body are $P'(x + u, y + v, z + w)$ and $Q'(x + u + \Delta x + \Delta u, y + v + \Delta v, z + w + \Delta w)$ as shown in Figure 158(b). We let $P'Q'$ be a principal diagonal to a rectangular parallelepiped with faces parallel to the axes as shown in Figure 158(b). We label vertices $M(x + u + \Delta x + \Delta u, y + v, z + w)$ and $N(x + u + \Delta x + \Delta u, y + v + \Delta v, z + w)$ as shown. The angle $MP'N$ made by the horizontal projection $P'N$ of $P'Q'$ and the positive half of the $x$-axis is designated by $\alpha$, measured from $P'M$. The directed distance $P'M$ is $(\Delta x + \Delta u)$, and the directed distance $MN$ is $\Delta v$. The relative elongation in the $x$-direction is



(a) Undeformed (b) Deformed

Fig. 158. Points in the body $B$.

$$\frac{P'M - PQ}{PQ},$$

which is $\Delta u/\Delta x$. Taking the limit as $\Delta x \to 0$ we obtain $\partial u/\partial x$, that is, $\epsilon_x$, as the strain in the $x$-direction. It follows similarly that $\partial v/\partial y$ and $\partial w/\partial z$ are the strains in the $y$- and $z$-directions respectively. They are called *longitudinal strains.*

We shall give simple geometric interpretations of the other first partial derivatives of $u$, $v$, and $w$. From Figure 158(b) we have

$$\tan \alpha = \frac{MN}{P'M} = \frac{\Delta v}{\Delta x + \Delta u}.$$

Taking the limit as $\Delta x$ goes to zero we have

$$\tan \alpha_0 = \frac{\dfrac{\partial v}{\partial x}}{1 + \dfrac{\partial u}{\partial x}},$$

where $\alpha_0$ is the limit of $\alpha$ as $\Delta x$ goes to zero. The angle $\alpha_0$ is thus the angle of inclination of the horizontal projection of the tangent line to the curve $P'Q'$ at $P'$ with respect to the $x$-axis. Assuming that $\partial u/\partial x$ is small compared to 1, as is generally true in practice, we have

$$\tan \alpha_0 = \frac{\partial v}{\partial x}.$$

Figure 158(b) is a special one, but the argument is perfectly general.

Let $PR$ be a straight line segment parallel to the $y$-axis in the undeformed body of Figure 158(a), and $P'R'$ the corresponding curve in the deformed body of Figure 158(b).

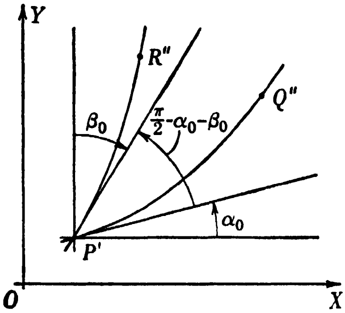


**Fig. 159. Horizontal projections of curves in the deformed body *B*.**

The derivative $\partial u/\partial y$ is now the tangent of the angle inclination $\beta_0$ of the tangent line to $P'R''$ with respect to the $y$-axis, where $P'R''$ is the horizontal projection of the curve $P'R'$. In Figure 159 we have pictured the horizontal projection $P'Q''$ of the curve $P'Q'$ as well as the horizontal projection $P'R''$ of $P'R'$. Here we have written $Q''$ for the $N$ of Figure 158(b). The right angle $QPR$ deforms into the angle $\left(\frac{\pi}{2} - \alpha_0 - \beta_0\right)$ indicated in Figure 159. For small angles $\alpha_0$ and $\beta_0$ we can replace $\alpha_0$ and $\beta_0$ by their tangents, so that the change in $QPR$ is $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, that is, the *shear strain* $\gamma_{xy}$. The term *shear* is used because of physical considerations which become clearer when the corresponding stresses are studied.

Because the strains (9.10) involve relative elongations, they are commonly referred to as *unit strains*. Thus $\epsilon_x$ is a unit elongation, and $\gamma_{xy}$ a unit shear strain.

We have shown how with the aid of (9.8) we can compute the unit elongation $\epsilon$ in any direction at a point $P$ in the body $B$ when the six components of strain (9.10) are given. As we shall show, the change in any angles at $P$ between two curves through $P$ (assumed to have tangent lines at $P$) is also determined by the components of strain. Let $C_1$ and $C_2$ be two curves of the undeformed body $B$ whose tangent lines at $P$ have the direction cosines $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$ respectively. Let $C_1'$ and $C_2'$ be the corresponding curves in the deformed body, and let their corresponding direction cosines be denoted by $l_1'$, $m_1'$, $n_1'$ and $l_2'$, $m_2'$, $n_2'$ respectively. By the relations of type (9.7) the direction cosines in the sets $l_1'$, $m_1'$, $n_1'$ and $l_2'$, $m_2'$, $n_2'$ are certain combinations of $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$ respectively. It is readily seen that the angle $\theta'$ between $C_1'$ and $C_2'$, corresponding to the direction cosines given above, is determined by the $l$'s, $m$'s, and $n$'s for $C_1$ and $C_2$, and the strain components at $P$. The change $(\theta' - \theta)$ in the angle $\theta$ between the curves $C_1$ and $C_2$ at $P$ can thus be found from the directions of $C_1$ and $C_2$ at $P$ and the strain components at $P$ regardless of what curves $C_1$ and $C_2$ are chosen at $P$, provided that they have tangent lines at $P$.

**9.5. Relations between stresses.** We shall prove that the shear stresses are dependent, in fact

$$\tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}, \tag{9.14}$$

whence the stress tensor is symmetric.

To prove (9.14) we refer to Figure 155, where we have an element in the form of a rectangular parallelepiped with edges $\Delta x$, $\Delta y$, and $\Delta z$, and with three faces on the coordinate planes, the other faces being in the first octant. We shall compute the moment of the forces acting on the element where this moment is taken with respect to the $x$-axis. Assuming that the element is in equilibrium under these forces (include dynamic equilibrium), it follows that the total moment with respect to the $x$-axis must vanish.

We emphasize that in the notation $\tau_{xy}$ the subscript $y$ refers to the axis to which the stress represented by $\tau_{xy}$ is parallel, and the subscript $x$, that is, the first subscript, denotes the direction of the normal to the plane at which the stress acts. The positive directions of the shearing stresses are in the positive directions of the axes for the faces of the element in the first octant not on the coordinate planes, and just the opposite for the faces which lie in the coordinate planes. If the coordinate axes are translated, the positive directions of the stresses remain the same. It is no restriction on the generality of the method to take the axes at the element as shown.

If a rectangular parallelepiped element has its faces parallel to the coordinate planes, the faces whose points have the more positive coordinates will be called *positive faces*, and the other faces *negative*. It follows that the positive directions of the shearing stresses are the same as the positive directions of the coordinate axes for points on positive faces, and the negative directions of the axes for negative faces.

We shall use primes on the symbols for the stresses on the planes of the element that are not coordinate planes, as in Figure 155.

In Section 2.11 we defined the moment of a force with respect to a point $O$. The definition can be extended in a simple manner to the moment of a force $F$ about a line as follows. Let this line be the $x$-axis. The force $F$ is the sum $(F_x + F_p)$ of the component of $F$ in the $x$-direction, and a component $F_p$ perpendicular to the $x$-axis. Suppose that $F_p \neq 0$. Let $A$ be the point on the $x$-axis where the $x$-axis cuts the plane through the line of $F_p$ perpendicular to the $x$-axis. The moment $F$ with respect to the $x$-axis is the moment of the perpendicular component $F_p$ with respect to the point $A$. If $F$ tends to rotate a body $B$ counterclockwise about the $x$-axis as seen from the positive end of the $x$-axis, the moment of $F$ with respect to this axis is positive; otherwise it is negative. If $F_p$ is zero, the corresponding moment is zero.

We suppose that the stresses vary continuously on and in the body $B$ of which an element is pictured in Figure 155. Let $P_1$ be a point on the bottom face of the element in this figure, and let $\epsilon_1\Delta y$ be the $y$-coordinate of $P_1$ for a number $\epsilon_1$ for which $0 \leqq \epsilon_1 \leqq 1$. By the first theorem of the mean for integrals we can choose $P_1$ so that

$$-(\epsilon_1\Delta y)\sigma_z\Delta x\Delta y$$

is the moment with respect to the $x$-axis of the force due to the stress component $\sigma_z$ on the bottom face of the element. Here $\sigma_z$ is understood to be evaluated at $P_1$.

Let $P_2$ be a point on the top face of the element, and let $\epsilon_2\Delta y$ be the $y$-coordinate of $P_2$ for a number $\epsilon_2$ with $0 \leqq \epsilon_2 \leqq 1$. We can choose $P_2$ so that

$$(\epsilon_2\Delta y)\sigma_z'\Delta x\Delta y$$

is the moment with respect to the $x$-axis of the force due to the stress represented by $\sigma_z'$ on the top face of the element, the function $\sigma_z'$ being evaluated at $P_2$.

Assuming the existence of enough derivatives, we have

$$\sigma_z' = \sigma_z + \frac{\partial \sigma_z}{\partial x}\,\delta\Delta x + \frac{\partial \sigma_z}{\partial y}\,\delta\Delta y + \frac{\partial \sigma_z}{\partial z}\,\Delta z + \eta_1,$$

where $\delta\Delta x$, $\delta\Delta y$, and $\Delta z$ are the coordinates of $P_2$ minus the corresponding coordinates of $P_1$, and $\eta_1$ is a quantity that will contribute nothing to the final limit in the process to follow. It follows that the moment $M_1$ (with respect to $OX$) corresponding to the quantities $\sigma_z$ and $\sigma_z'$ can be written as

$$M_1 = \Delta x(\Delta y)^2\left[(\epsilon_2 - \epsilon_1)\sigma_z + \epsilon_2\left(\frac{\partial \sigma_z}{\partial x}\,\delta\Delta x + \frac{\partial \sigma_z}{\partial y}\,\delta\Delta y + \frac{\partial \sigma_z}{\partial z}\,\Delta z + \eta_1\right)\right]. \tag{9.15}$$

As $\Delta z$ goes to 0, we have $\epsilon_2 \to \epsilon_1$, $\delta\Delta x \to 0$, $\delta\Delta y \to 0$, $\eta_1 \to 0$, whence the quantity in the brackets goes to zero with $\Delta z$. Thus $M_1$ goes to zero with $\Delta z$.

The same type of treatment applies to the moments with respect to $OX$ corresponding to $\sigma_y$, $\sigma'_y$, as well as the moments for shear stresses represented by $\tau_{xy}$, $\tau'_{xy}$, and $\tau_{xz}$, $\tau'_{xz}$. There remain the stresses represented by $\tau'_{yz}$ and $\tau'_{zy}$. These contribute the moment $M_2$ given by

$$M_2 = \Delta y(\tau'_{yz}\Delta x\Delta z) - \Delta z(\tau'_{zy}\Delta x\Delta y), \tag{9.16}$$

where $\tau'_{yz}$ and $\tau'_{zy}$ are evaluated at properly chosen points.

The body $B$ may be influenced by a field of force whose intensity has the $x$, $y$, and $z$ components represented by $F_x$, $F_y$, and $F_z$ respectively. Assuming that these components are continuous functions of $x$, $y$, and $z$ over the element, we have the resulting moment $M$ with respect to the $x$-axis given by

$$M = (\epsilon_3\Delta yF_z - \epsilon_4\Delta zF_y)\Delta x\Delta y\Delta z \tag{9.17}$$

for $\epsilon$'s in the range 0 to 1, and functions $F_y$, $F_z$ evaluated at points of the element. We can include *inertial forces* in $M$.

It follows from (9.15), (9.16), and (9.17) that if we let $\Delta x = \Delta y = \Delta z$, divide the sum of the moments with respect to the $x$-axis by $(\Delta x)^3$, and take the limit as $\Delta x \to 0$, we obtain

$$\tau_{yz} = \tau_{zy} \tag{9.18}$$

valid at $O$, whence the stress tensor is symmetric.

**9.6. Stress on arbitrary plane.** In Section 9.3 we showed that the six strain components $\epsilon_x$, $\epsilon_y$, $\epsilon_z$, $\gamma_{xy}$, $\gamma_{zx}$, $\gamma_{yz}$ are sufficient to determine the strain in any direction at the given point $P$, so that in this sense these components form a complete set. We shall prove similarly for the six components of stress $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$, $\tau_{yz}$, $\tau_{zx}$ that the stress on any plane through a given point $P$ of the body $B$ is determined by the six components of stress at $P$.

Consider the tetrahedron $ABCP$ of Figure 160 bounded by planes parallel to the coordinate planes and a plane $\pi$ whose outward normal has the direction cosines $l$, $m$, $n$ as shown. The edges $AP$, $BP$, and $CP$ are parallel to the $x$-, $y$-, and $z$-axes respectively. We let $T$ designate the stress at a point on $\pi$, and let $p$ be the perpendicular distance from $P$ to the plane $\pi$. The shear stresses will be denoted without primes, with single primes, and with double primes on the faces $BCP$, $ABP$, and $APC$ respectively. We suppose first that $l$, $m$, and $n$ are all different from zero, that is the plane $\pi$ is not parallel to one of the axes. We let $a$ be the area of the part $ABC$ of $\pi$ which bounds the tetrahedron, and represent the components of the body force intensity by $F_x$, $F_y$, and $F_z$, respectively. The area of the bottom face

*ABP* is *an*. Similar formulas hold for the other faces. We let $T_x$, $T_y$, and $T_z$ represent the components of the stress $T$ (hence surface force intensity) in the $x$-, $y$-, and $z$-directions respectively. The $x$-component of force acting on the face *ABC* is represented by $T_x a$ for $T_x$ evaluated at a properly chosen point on this face. We assume continuity of all components involved in the argument. Due to the stress represented by $\sigma_x$ on the face *PBC* the quantity $-\sigma_x al$ represents a force component in the $x$-direction for $\sigma_x$, evaluated at a properly chosen point. From the face *APC* we have a force component represented by $-\tau''_{yx}am$, and from *ABP* a force component represented by $-\tau'_{zx}an$. We also have a body force component represented by $(F_x ap)/3$ for $F_x$ evaluated at a properly chosen point of the tetrahedron. We include *inertial* forces in $F_x$, $F_y$, $F_z$ to take care of dynamic equilibrium. Since the sum of the $x$-components of force on the element vanish, we have

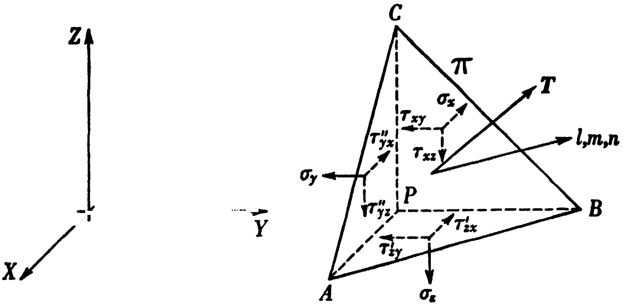


Fig. 160. Stress $T$ on a plane $\pi$.

$$T_x a - \sigma_x al - \tau''_{yx}am - \tau'_{zx}an + \frac{(F_x ap)}{3} = 0.$$

Dividing by $a$ and taking the limit as $p$ goes to zero while $l$, $m$, and $n$ remain constant we derive

$$T_x = l\sigma_x + m\tau_{yx} + n\tau_{zx} \tag{9.19}$$

valid at $P$. Similarly,

$$T_y = l\tau_{xy} + m\sigma_y + n\tau_{zy}, \tag{9.20}$$

$$T_z = l\tau_{xz} + m\tau_{yz} + n\sigma_z. \tag{9.21}$$

The ± signs of the force intensities $T_x$, $T_y$, $T_z$ are taken so that the sign is positive if the intensity has the positive sense of the corresponding axis. This is not the convention for stresses. Thus if $l = -1$, $m = n = 0$, formula (9.19) yields $T_x = -\sigma_x$, when the normal stress component in the $x$-direction is $\sigma_x$.

The above derivation of (9.19)–(9.21) is based on the assumption that $l$, $m$, $n$ are all different from zero. Continuity assumptions yield the validity of (9.19)–(9.21) when at least one of the direction cosines $l$, $m$, $n$ vanishes.

We proved (9.19)–(9.21) by using a tetrahedron in the first octant. The treatment is easily seen to be valid in the general case.

The quantities $\sigma_x$, $\tau_{yx}$, $\tau_{zx}$ represent the $x$-, $y$-, and $z$-components of the stress on a plane parallel to the $yz$-plane. Thus the stresses at a point $P$ on any plane can be expressed in terms of the stresses on three mutually perpendicular planes through $P$.

Let $S$ be a surface through the point $P$, and $\pi_S$ the tangent plane to $S$ at $P$. Assume that a normal ray to $S$ at $P$ exists. From the manner in which $\pi_S$ approximates $S$ in a neighborhood of $P$ the stress on $S$ at $P$ is easily seen to be the same as the stress on $\pi_S$ at $P$, in fact the stress on $S$ depends only on the direction of the normal ray to $S$ at $P$ and the stresses on three mutually perpendicular planes through $P$. There are two directions of the normal rays to $\pi_S$ corresponding to the two different sets of directions cosines of the normal lines to $\pi_S$. Let $B_1$ be the part of the body on the same side of $S$ as the normal ray to $S$ at $P$ and $B_2$ the part of the body on the other side. It is convenient to think of the stress on $S$ as due to the action of $B_1$ on $B_2$, at least in a sufficiently small neighborhood of $P$.

It follows from (9.19)–(9.21) that the stresses on any plane through $P$ are determined by the six components of stress $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$, $\tau_{yz}$, $\tau_{zx}$.

**9.7. Stress-strain relations.** For an elastic body the strains are functions of the stresses, so that

$$\epsilon_x = f(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}),$$

and similarly for the other strains. It is reasonable to suppose that $f(\sigma_x, \ldots, \tau_{zx})$ can be expanded as a power series in $\sigma_x, \ldots, \tau_{zx}$. For this series the constant term vanishes inasmuch as $\epsilon_x = 0$ when the stresses vanish. We suppose that the series converges for the stresses under consideration. This is reasonable because in a given problem we often have the stresses initially zero and we would expect the series to converge for some range of values of the stress components near zero. Unless the stresses are large we can expect to be able to discard all but the linear terms in the expansion so that

$$\epsilon_x = a\sigma_x + b\sigma_y + c\sigma_z + d\tau_{xy} + e\tau_{zx} + f\tau_{yz} \tag{9.22}$$

with similar relations valid for the other strains, the coefficients of the stress components being real numbers. *We should always assume that linear relations like (9.22) hold unless there is good evidence to the contrary.* For even such a nonuniform substance as wood it is customary to suppose that linear relations hold between the stress and strain components.

In the manufacture of many items, such as steel parts, an effort is generally made to have the material uniform, that is, homogeneous and isotropic. By *isotropic* we mean here that the elastic properties at a point are the same for each direction through the point. *Homogeneous* designates that the elastic properties at different points are the same. In this event the linear relations between the stresses and strains are quite simple as we shall show. Most structural materials are crystalline. However, the crystals are generally small compared with the dimensions of the material used, and chaotically distributed, so that the assumption of homogeneity and isotropy is a good one. Wood is not isotropic but generally has three mutually orthogonal planes of elastic symmetry. We assume in the rest of this section that the body $B$ is isotropic and homogeneous.

Since the $y$- and $z$-axes are individually related to the $x$-axis in the same way, it follows that

$$b = c,\ d = e,$$

whence by (9.22)

$$\begin{aligned}\epsilon_x &= a\sigma_x + b(\sigma_y + \sigma_z) + d(\tau_{xy} + \tau_{zx}) + f\tau_{yz},\\ \epsilon_y &= a\sigma_y + b(\sigma_x + \sigma_z) + d(\tau_{xy} + \tau_{yz}) + f\tau_{zx},\\ \epsilon_z &= a\sigma_z + b(\sigma_x + \sigma_y) + d(\tau_{yz} + \tau_{zx}) + f\tau_{xy}.\end{aligned} \tag{9.23}$$

We can similarly write the shear strains as

$$\begin{aligned}\gamma_{xy} &= g(\sigma_x + \sigma_y) + h\sigma_z + q\tau_{xy} + p(\tau_{zx} + \tau_{yz}),\\ \gamma_{yz} &= g(\sigma_y + \sigma_z) + h\sigma_x + q\tau_{yz} + p(\tau_{zx} + \tau_{xy}),\\ \gamma_{zx} &= g(\sigma_z + \sigma_x) + h\sigma_y + q\tau_{zx} + p(\tau_{xy} + \tau_{yz}).\end{aligned} \tag{9.24}$$

We introduce an $x'$-axis as shown in Figure 161, which axis is the same as the $x$-axis with the sense reversed, so that $x' = -x$. In this figure we

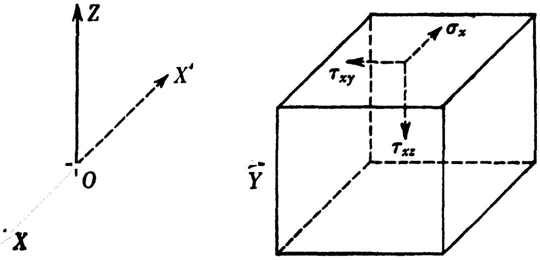


**Fig. 161. New $x'$-axis.**

have a rectangular parallelepiped element with faces parallel to the axes. Since $B$ is homogeneous and isotropic the stress-strain relations must be the same for any Cartesian coordinate system, and in particular these relations for the $yx'z$-coordinate system must be the same as these relations for the $xyz$-coordinate system. The negative faces on which the stresses represented by $\tau_{xy}$ and $\tau_{xz}$ act in Figure 161 for the $xyz$-coordinate system are positive faces for the $yx'z$-coordinate system. It follows that the positive directions of the shear stresses represented by $\tau^*_{x'y}$, $\tau^*_{x'z}$ on these faces for the $yx'z$-coordinate system are opposite to the positive directions of the stresses represented by $\tau_{xy}$ and $\tau_{xz}$ respectively, so that

$$\tau^*_{x'y} = -\tau_{xy}, \tau^*_{x'z} = -\tau_{xz}.$$

Examination of the stresses indicated in Figure 155 shows that the other stresses with respect to the $xyz$-coordinate system go over without change in sign. We have thus

$$\sigma^*_{x'} = \sigma_x, \sigma^*_y = \sigma_y, \sigma^*_z = \sigma_z, \tau^*_{x'y} = -\tau_{xy}, \tau^*_{zx'} = -\tau_{zx}, \tau^*_{yz} = \tau_{yz}, \tag{9.25}$$

where we use asterisks to represent stresses in the $yx'z$-coordinate system. Asterisks will similarly be used for strains. Since the lines of the axes are not changed in going from the $xyz$-coordinate system to the $yx'z$ coordinate system the unit elongations are unchanged. Let $AB$, $AC$, and $AB'$ be rays in the directions of $OX$, $OY$, and $OX'$ respectively, so that angles $BAC$ and $B'AC$ are orientated in the same way as $XOY$ and $X'OY$ respectively. The angle $B'AC$ is the supplement of the angle $BAC$, whence an increase (or decrease) in $BAC$ corresponds to a decrease (or increase) by the same amount in $B'AC$. The same applies to angles oriented like $XOZ$ and $X'OZ$.

We have

$$\epsilon^*_{x'} = \epsilon_x, \epsilon^*_y = \epsilon_y, \epsilon^*_z = \epsilon_z, \gamma^*_{x'y} = -\gamma_{xy}, \gamma^*_{zx'} = -\gamma_{zx}, \gamma^*_{yz} = \gamma_{yz}. \tag{9.26}$$

By (9.23) we have

$$\epsilon^*_y = a\sigma^*_y + b(\sigma^*_{x'} + \sigma^*_z) + d(\tau^*_{x'y} + \tau^*_{yz}) + f\tau^*_{zx'}.$$

By (9.25) and (9.26) this yields

$$\epsilon_y = a\sigma_y + b(\sigma_x + \sigma_z) + d(-\tau_{xy} + \tau_{yz}) - f\tau_{zx}. \tag{9.27}$$

From later considerations of this chapter it will appear that we can have any one of the stresses not zero while the other stresses vanish, in fact, except for physical limitations we can have any value of any one stress while the other stresses vanish. It follows that the formulas (9.27) and (9.23) for $\epsilon_y$ must be identical in the stresses, whence

$$d = f = 0.$$

For $\gamma^*_{x'y}$ we have by (9.24)

$$\gamma^*_{x'y} = g(\sigma^*_{x'} + \sigma^*_y) + h\sigma^*_z + q\tau^*_{x'y} + p(\tau^*_{zx'} + \tau^*_{yz}).$$

Substituting from (9.25) and (9.26) we derive the relations

$$-\gamma_{xy} = g(\sigma_x + \sigma_y) + h\sigma_z - q\tau_{xy} + p(-\tau_{zx} + \tau_{yz}).$$

Comparison with the formula for $\gamma_{xy}$ in (9.24) yields

$$g = h = p = 0.$$

The relations (9.23) and (9.24) now become

$$\begin{aligned} \epsilon_x &= a\sigma_x + b(\sigma_y + \sigma_z), & \gamma_{xy} &= q\tau_{xy}, \\ \epsilon_y &= a\sigma_y + b(\sigma_x + \sigma_z), & \gamma_{yz} &= q\tau_{yz}, \\ \epsilon_z &= a\sigma_z + b(\sigma_x + \sigma_y), & \gamma_{zx} &= q\tau_{zx}. \end{aligned} \tag{9.28}$$

We know from experience that relations (9.28) cannot be simplified by removal of other terms. In fact a tension in the $x$-direction causes an elongation in the $x$-direction and contractions in the $y$- and $z$-directions, whence $a$ and $b$ are in general different from zero. As will be seen below we can produce a pure shear $\tau_{xy}$ that causes a corresponding strain in the body, whence $q \neq 0$. The constants $a$, $b$, and $q$ are dependent as will now be shown.

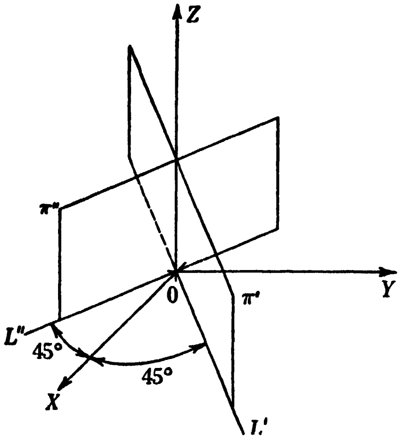


Fig. 162. Planes making 45° with the $xz$-plane.

We let the origin of the coordinate system be chosen so that this is the point at which the stresses are being taken. We consider a long beam (infinitely long to be mathematically rigorous) with the $z$-axis as axis, and constant horizontal cross section in the form of a rectangle with sides parallel to the $x$ and $y$-axes. We assume that the following set of stresses apply to the beam

$$\sigma_x = -\sigma_y, \; \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0,$$

and that $\sigma_x$ is constant. The reader can verify that these stresses satisfy the fundamental equations of elasticity, not all yet derived. These stresses correspond to tension on the beam in one direction and compression in the other.

The $z$-coordinate is now out of the picture, there being no strains involving $z$. We pass planes $\pi'$ and $\pi''$ through the $z$-axis as in Figure 162 making 45 degrees with the $xz$-plane. The plane $\pi'$ cuts into the first octant. We let $l'$, $m'$, $n'$, and $l''$, $m''$, $n''$ be direction cosines of normals to the planes $\pi'$ and $\pi''$ respectively. It follows that we can take

$$l' = -m' = \frac{\sqrt{2}}{2},\ n' = 0;\ l'' = m'' = \frac{\sqrt{2}}{2},\ n'' = 0.$$

Let primes on notations for surface force intensities designate intensities on the plane $\pi'$, and double primes intensities on $\pi''$. By (9.19) and (9.20)

$$T'_x = \frac{\sigma_x}{\sqrt{2}},\ T'_y = \frac{\sigma_x}{\sqrt{2}}.$$

It appears from Figure 163 that the force intensity components represented by $T'_x$ and $T'_y$ yield a shear stress on $\pi'$ represented by $\tau$, where $\tau = \sigma_x$. By (9.19) and (9.20)

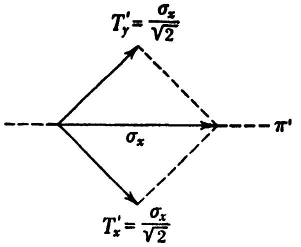

$T'_y = \frac{\sigma_x}{\sqrt{2}}$
$\pi'$
$\sigma_x$
$T'_x = \frac{\sigma_x}{\sqrt{2}}$

**Fig. 163. Stress on a plane $\pi'$.**

$$T''_x = \frac{\sigma_x}{\sqrt{2}},\ T''_y = \frac{-\sigma_x}{\sqrt{2}},$$

whence there is a shear stress on $\pi''$ represented by $\tau$, where $\tau = \sigma_x$. Since the stresses on all planes through $O$ are determined by the stresses on three mutually perpendicular planes through $O$, it follows that the stresses at $O$ are determined by the shearing stresses represented by $\{\tau\}$ on $\pi'$ and $\pi''$, there being no stress on the $xy$-plane. We have a case of *pure shear.* It follows that $q$ is a function of $a$ and $b$.

By (9.28) we now have the strains

$$\epsilon_x = (a - b)\sigma_x,\quad \epsilon_y = (b - a)\sigma_x,$$

whence by integration

$$u = x(a - b)\sigma_x + f(y),\quad v = y(b - a)\sigma_x + g(x)$$

for functions $f(y)$ and $g(x)$. Due to symmetry the points originally on the $x$- and $y$-axes remain on their respective axes after deformation. Thus when $x = 0$, we have $u = 0$ also, whence $f(y) = 0$. Similarly $g(x) = 0$.

By (9.28) the shear stress represented by $\tau$ causes no elongations in the directions of the lines $L'$ and $L''$ where the planes $\pi'$ and $\pi''$ cut the

$xy$-plane. The derivative $ds/ds'$ in (9.7), relating arc lengths on the undeformed and deformed curves, is thus unity for each of the lines $L'$ and $L''$. Let $l_1'$, $m_1'$, $n_1'$ and $l_1''$, $m_1''$, $n_1''$ be the direction cosines of the tangent lines to the respective curves into which $L'$ and $L''$ are deformed by the stress distribution. By (9.7)

$$\begin{aligned} l_1' &= [1 + (a - b)\sigma_x]\frac{\sqrt{2}}{2}, \\ m_1' &= [1 + (b - a)\sigma_x]\frac{\sqrt{2}}{2}, \\ l_1'' &= [1 + (a - b)\sigma_x]\frac{\sqrt{2}}{2}, \\ m_1'' &= -[1 + (b - a)\sigma_x]\frac{\sqrt{2}}{2}, \\ n_1' &= n_1'' = 0. \end{aligned} \tag{9.29}$$

Let $\gamma$ represent the shearing strain for the lines $L'$ and $L''$. Further, let $\theta$ be the angle into which the angle $L'OL''$ of Figure 162 deforms under the given stresses. By (9.29) we have

$$\cos\theta = l_1'l_1'' + m_1'm_1'' + n_1n_1'' = 2(a - b)\sigma_x.$$

The shearing strain for $L'$ and $L''$ is $\frac{\pi}{2} - \theta$. Assuming that $\frac{\pi}{2} - \theta$ is small we can write

$$\frac{\pi}{2} - \theta = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta.$$

It follows that the shearing stress $\gamma$ is given by

$$\gamma = 2(a - b)\sigma_x = 2(a - b)\tau,$$

whence

$$q = 2(a - b)$$

as was to be proved.

*Hooke's law* states that with $\sigma_x$ as the only stress

$$\epsilon_x = \frac{\sigma_x}{E} \tag{9.30}$$

for the *modulus E of elasticity in tension*, whence $a = 1/E$. For steel $E$ is about 30,000,000 pounds per square inch. Since a typical value of the elastic limit for steel is around 30,000 pounds per square inch, the strain $\epsilon_x$ in the case of steel is generally smaller than 0.1 per cent. The elastic properties hold up to the elastic limit. For rubber a typical value of $E$ is 300 pounds per square inch, and as is well known, the strains can be relatively large. The law (9.30) holds for both compression and tension.

It is customary to write

$$b = -\frac{\mu}{E},$$

where $\mu$ is *Poisson's ratio.* Since from experiment we know that extensions are accompanied by lateral contractions, we expect $b$ to be negative, which is the case. Typical values of $\mu$ are 0.244 for glass, 0.333 for copper, 0.303 for steel, and 0.15 for concrete. A rough average value of $\mu$ is thus $\frac{1}{4}$.

In terms of $\mu$ we have

$$q = \frac{2(1 + \mu)}{E}.$$

It is customary to write the relations (9.28) as

$$\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z), & \gamma_{xy} &= \frac{\tau_{xy}}{G}, \\ \epsilon_y &= \frac{\sigma_y}{E} - \frac{\mu}{E}(\sigma_x + \sigma_z), & \gamma_{yz} &= \frac{\tau_{yz}}{G}, \\ \epsilon_z &= \frac{\sigma_z}{E} - \frac{\mu}{E}(\sigma_x + \sigma_y), & \gamma_{zx} &= \frac{\tau_{zx}}{G} \end{aligned} \tag{9.31}$$

for

$$G = \frac{E}{2(1 + \mu)}.$$

The quantity $G$ is termed the *modulus of elasticity in shear*, or *modulus of rigidity.*

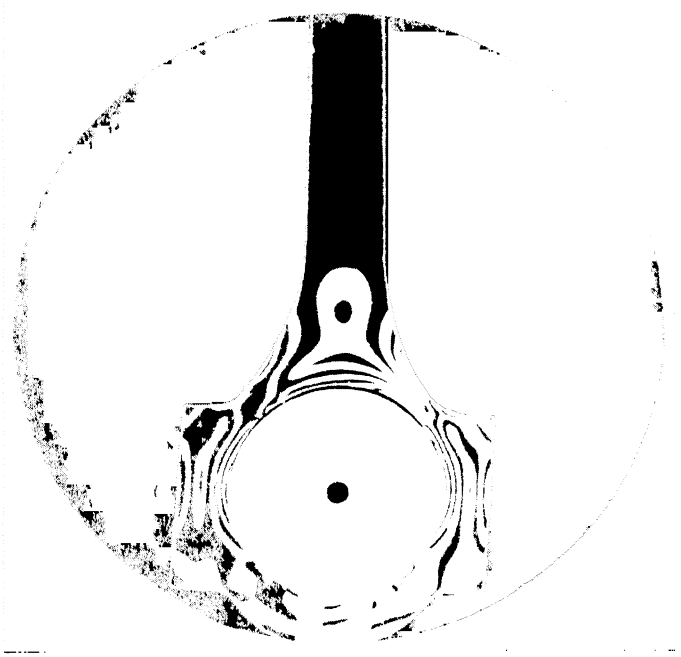


**Fig. 164. A connecting rod model under stress photographed with polarized light. Courtesy of the photoelasticity laboratory of the Chrysler Corporation.**

From the equilibrium of forces on subbodies of the body $B$ we can obtain three conditions on the stresses and body forces corresponding to the vanishing of the total force components in the directions of the coordinate axes. These three conditions, with (9.10) and (9.31), yield nine conditions on the six stress components and the displacements $u$, $v$, and $w$. These conditions and the stresses on the boundary of $B$ or the displacements on the boundary of $B$, given also the body forces, are often enough to determine the state of stress of the body $B$ for rather general conditions, as can be shown by the uniqueness theory of partial differential equations. The conditions obtained from the equilibrium of forces will be derived later for the two-dimensional case.

Generally, the uniqueness of the solution of the nine conditions on the six stress components and three displacements is proved (physically) by assuming that the body $B$ is a thermodynamic system to which the first and second laws of thermodynamics apply. For two dimensions uniqueness can often be readily shown where stress functions exist (see reference 2).

### 9.8. Plastic flow and creep.

For *plastic deformation* the rate of change of the strain with respect to time is a function of the stress. With $v_x$ denoting the derivative $\partial\epsilon_x/\partial t$, Hooke's law (9.30) is replaced by

$$v_x = f(\sigma_x)$$

for a function $f$. From experience we know that there is no change in the deformation when the stress is zero, whence it is reasonable to suppose that

$$v_x = k\sigma_x^a \tag{9.32}$$

for constants $a$ and $k$. This approximation is used successfully in practice.

In the case of *creep* the rate of deformation decreases with time. Here we can use the law (9.32) with $k$ decreasing with time. It is evident that all kinds of relationships between stresses and strains are theoretically possible. Experimental observation of the nature of the changes of state of the body under consideration should suggest reasonable relations between stresses and strains, as well as other variables.

The terms "plastic deformation" and "creep" are sometimes associated with other stress-strain relations.

## REFERENCES

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2. Ivan S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw-Hill, New York, 1946.
3. S. Timoshenko, *Theory of Elasticity*, McGraw-Hill, New York, 1934.

# CHAPTER

# 10

# *Elasticity for one and two dimensions*

**10.1. Stretched wire.** The stress-strain relations derived in previous sections are generally too complicated to apply directly in practice. We should therefore replace the given body or bodies by the system that is as simple as possible geometrically and will still give satisfactory results. We should idealize the bodies by equivalent ones of as low dimensions as possible, and of the simplest shapes. Thus we may be able to replace a body of elliptical cross section by one of circular cross section. We shall illustrate by beginning with wires having a cross section normally so small in comparison with their length that, in what follows, the cross section will be considered as a point.

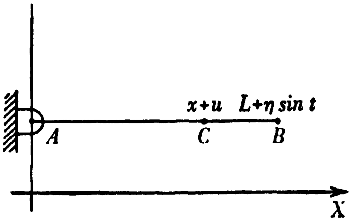


**Fig. 165. Stretched wire.**

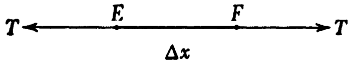


**Fig. 166. Section of a stretched wire.**

Consider a wire fixed at one end $A$, and stretched out so as to lie at all times on the same straight line. Suppose that the other end $B$ is moved sinusoidally along the line. The reader is referred to Figure 165, where the coordinate of $B$ is $(L + \eta \sin t)$. In practice $\eta$ would be small compared to $L$. If the point $C$ has the abscissa $x$ when $t = 0$, this point has the abscissa $(x + u)$ at the time $t$ for the displacement $u$ of $C$. We let $T$ represent the force due to tension. The problem is to determine $T$ and $u$.

We suppose that the mass of the wire can be neglected. In Figure 166 we have pictured a section $EF$ of the wire of length $\Delta x$. From the vanishing of forces on the segment it follows that the tension force at the end $E$ is the same as the tension force at $F$ but oppositely directed. As a consequence the tension along the wire at any given instant is constant. By

Hooke's law, assuming that $\partial u/\partial x$ exists,

$$\frac{\partial u}{\partial x} = kT(t) \tag{10.1}$$

for a constant $k$, and the tension $T(t)$. With the aid of boundary conditions the solution is now determined. The boundary conditions to be used will be found from the solution.

Integrating (10.1) we obtain the formula

$$u = kxT(t) + g(t)$$

for a function $g(t)$ of $t$. At $x = L$ we have $u = \eta \sin t$. It follows that

$$u = k(x - L)T(t) + \eta \sin t. \tag{10.2}$$

Since the point $A$ is fixed we have $u = 0$ when $x = 0$. By (10.2)

$$T(t) = \frac{\eta \sin t}{kL},$$

whence the tension at each instant is known. By this formula for $T(t)$ and (10.2) the displacement of each point on the wire is determined. We thus obtain the easily anticipated result that $u = (x/L)(\eta \sin t)$.

Where the tension is high enough we are justified in neglecting inertial and damping forces.

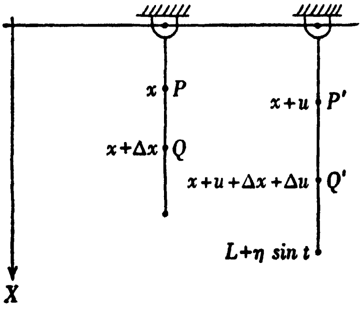


Fig. 167. Element of a vertical wire.

*The boundary and initial conditions needed to solve a problem are perhaps best determined in the general case by first setting up as many independent equations as there are variables to be solved for. In the solution of these equations the initial and boundary conditions needed to carry through the solution will become evident.* Through the theory of the existence of solutions it is possible to state fairly comprehensive theorems on the initial and boundary conditions needed to solve various problems.

As a second example, we suppose that the wire is vertical, and that the tensions are small enough so that the mass of the wire cannot be neglected. In Figure 167 we have pictured an element $PQ$ of length $\Delta x$. We suppose that the weight is distributed uniformly along the wire when at rest. We let $\rho$ be the density of the wire per unit length, so that the element $PQ$ weighs $\rho g\Delta x$ units, inasmuch as the density is uniform. At an instant $t$ during the motion of the wire the points $P$ and $Q$ are at $P'$ and $Q'$ as shown in Figure 167. The coordinates $x$ and $(x + \Delta x)$ of $P$ and $Q$ have gone into $(x + u)$ and $(x + u + \Delta x + \Delta u)$ of $P'$ and $Q'$, respectively. Here $u$ is a function of $x$ and $t$. The density of the wire is in general no longer uniform. It will be denoted by $(\rho + \sigma)$. We assume that all functions and derivatives in our argument are continuous.

The mass of the wire section $PQ$ is identical with the mass in the section $P'Q'$. For some value of $(\rho + \sigma)$ between the minimum and maximum taken on by $(\rho + \sigma)$ along the segment $P'Q'$ the mass of the section $P'Q'$ is $(\rho + \sigma)(\Delta x + \Delta u)$. Equating masses yields

$$\rho\Delta x = (\rho + \sigma)(\Delta x + \Delta u).$$

Dividing by $\Delta x$ and letting $\Delta x$ go to zero gives

$$\rho = (\rho + \sigma)\left(1 + \frac{\partial u}{\partial x}\right).$$

If $\partial u/\partial x$ is small compared to 1, this gives $\rho = \rho + \sigma$, from which $\sigma = 0$, and $\rho + \sigma$ can be replaced by $\rho$.

To allow for varying density we must use Newton's force law $F = d(mv)/dt$ in the form

$$F = m\frac{dv}{dt} + v\frac{dm}{dt},$$

the velocity and mass being denoted by $v$ and $m$, respectively. We let $g$ be the magnitude of the acceleration of gravity, and $\Delta T$ the difference between the tension $(T + \Delta T)$ at $Q'$ and the tension $T$ at $P'$. As a consequence

$$(\rho + \sigma)(\Delta x + \Delta u)\frac{\partial^2(x + u + \eta\Delta x + \zeta\Delta u)}{\partial t^2}$$
$$+ \frac{\partial(x + u + \eta\Delta x + \zeta\Delta u)}{\partial t}\frac{\partial(\rho + \sigma)(\Delta x + \Delta u)}{\partial t} = g(\rho + \sigma)(\Delta x + \Delta u) + \Delta T$$

for $\eta$ and $\zeta$ between zero and 1, and factors evaluated at some properly chosen point with coordinate $(x + u + \eta\Delta x + \eta\Delta u)$ on the section $P'Q'$. Suppose that $\partial u/\partial x$ is small compared to 1. Note that $\Delta u$ is approximately $(\partial u/\partial x)\Delta x$. Dividing by $(\Delta x + \Delta u)$, taking the limit as $\Delta x$ goes to zero, (assuming that the limits exist), dropping $\partial u/\partial x$ in $1 + \partial u/\partial x$, and making use of the fact that $\rho$ and $x$ are independent of $t$, we have

$$(\rho + \sigma)\left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x\partial t} + \frac{\partial^2 u}{\partial t^2}\right) + \frac{\partial u}{\partial t}\frac{\partial \sigma}{\partial t} = g(\rho + \sigma) + \frac{\partial T}{\partial x}.$$

If $\partial\sigma/\partial t$ is sufficiently small we can write this equation as

$$(\rho + \sigma)\frac{\partial^2 u}{\partial t^2} = g(\rho + \sigma) + \frac{\partial T}{\partial x},$$

where we have dropped the term in the product of derivatives of $u$. By (10.1), remembering that $T$ is now a function of $x$ and $t$, and the assumption that $\sigma$ is small compared to $\rho$ we finally obtain

$$\frac{\partial^2 u}{\partial t^2} = g + \frac{1}{k\rho}\frac{\partial^2 u}{\partial x^2}. \tag{10.3}$$

To solve (10.3) for $u$ as a function of $x$ and $t$ we need the boundary conditions

$$x = 0,\ u = 0;\ x = L,\ u = \eta \sin t$$

and initial conditions such as $t = 0$, $u \equiv 0$; and the velocity of each point on the wire when $t = 0$.

If damping must be included, then a term like $-K(\partial u/\partial t)$ for a positive constant $K$ should be added to the right of (10.3).

**10.2. Flexible cable.** Chains, ropes, strings, and threads offer, for practical purposes, no resistance to bending, and will be called *flexible cables.* As in the case of wires the cross section is so small compared to the length that it will be neglected. Thus the flexible cables will be considered to be one-dimensional bodies, that is, simply curves in space with associated physical properties. A cable is generally supported at one or both ends, and hangs under the influence of gravity and other forces.

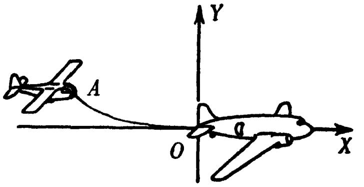


**Fig. 168. Airplane pulling a glider.**

Consider, for example, the case of an airplane pulling a glider as shown in Figure 168. The rope connecting the glider to the airplane joins the glider at $A$ and the airplane at $O$. The problem is to determine the stresses at points on the cable, and the geometry of the cable. It is important to know the stresses, because if they are too high the ultimate stress of the rope material will be exceeded, and the rope will break.

The speed of the airplane near the earth is normally high compared to the prevailing winds so that we may neglect these winds, and treat only the wind due to the motion of the airplane in still air, at least we consider this case only. We shall consider first the case of a steady state.

We set up $x$- and $y$-axes fixed to the airplane as shown in the figure. We shall suppose that the airplane is traveling in normal horizontal flight.

As a first approximation we neglect the stresses on the rope caused by the wind. We suppose that the rope is uniform, and weighs $w$ units per unit of length, and that the density can be considered constant for practical purposes. In Figure 169 we have pictured a section of the rope of length $\Delta s$ for arc length coordinate $s$. For purposes of explanation the rope section is taken below the $x$-axis in the figure. The coordinates $x$ and $y$ of a point $R$ on the rope are functions of $s$. The end points of the rope section are denoted by $a$ and $b$ respectively. We let $\theta$ be the angle of inclination of the tangent line to the rope at $a$, and $(\theta + \Delta\theta)$ the corresponding angle at the point $b$, where the angles of inclination are taken in the range $\frac{-\pi}{2}$ to $\frac{\pi}{2}$. We let $T$ designate the tension force at $a$, and $(T + \Delta T)$ the tension force at $b$. The force of gravity acting on the rope section is $w\Delta s$. The vanishing of the horizontal component of the total force on the section gives

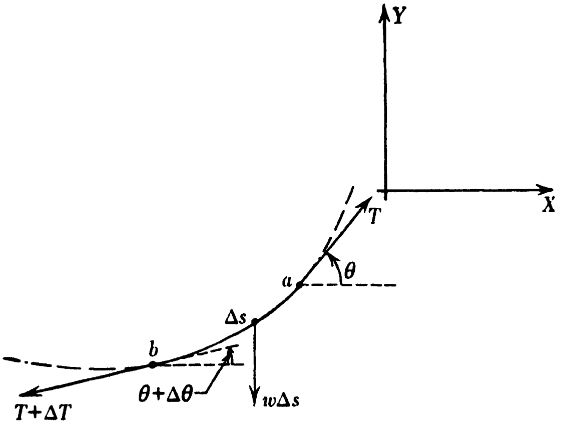


Fig. 169. Section of a glider rope.

$$(T + \Delta T)\cos(\theta + \Delta\theta) - T\cos\theta = 0.$$

Suppose that we measure arc length $s$ from $O$ so that $s = 0$ at $O$ and $s = L$ at the point $A$, where $L$ is the length of the stressed rope. The derivatives to be used in the argument are assumed to exist. Now

$$-\cos\theta = \frac{dx}{ds}.$$

We write

$$\Delta T = \frac{dT}{ds}\Delta s + \epsilon_1, \quad -\cos(\theta + \Delta\theta) = \frac{dx}{ds} + \frac{d^2x}{ds^2}\Delta s + \epsilon_2,$$

whence we have

$$T\left(\frac{d^2x}{ds^2}\Delta s + \epsilon_2\right) + \left(\frac{dT}{ds}\Delta s + \epsilon_1\right)\left(\frac{dx}{ds} + \frac{d^2x}{ds^2}\Delta s + \epsilon_2\right) = 0. \quad (10.4)$$

The quantities $\epsilon_1$ and $\epsilon_2$ will contribute nothing to the limit of the above expression after we have divided by $\Delta s$. This limit yields

$$T\frac{d^2x}{ds^2} + \frac{dT}{ds}\frac{dx}{ds} = 0. \quad (10.5)$$

Since the left side of this equation is the derivative of $T(dx/ds)$ with respect to $s$, it follows that

$$T\frac{dx}{ds} = K \quad (10.6)$$

for a constant $K$.

In the same manner the vanishing of the total force in the $y$-direction yields

$$\frac{d\left(T\frac{dy}{ds}\right)}{ds} = w.$$

Elimination of $T$ gives

$$\frac{d\frac{dy}{dx}}{ds} = \frac{w}{K} \quad (10.7)$$

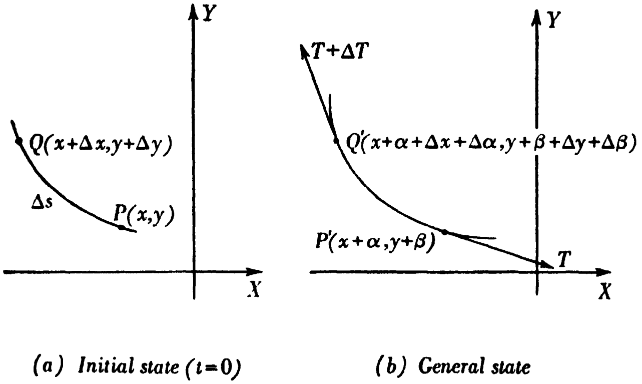
as the differential equation relating $x$ and $y$ along the rope. We can eliminate $ds$ from (10.7) and obtain a differential equation of the second order in $x$ and $y$.

The tension force acting on the glider at $A$ is now $K/(dx/ds)$ with the direction specified by $dy/dx$ at $s = L$. We must incorporate this force into the differential equations of motion of the glider, which can be obtained from that given for airplanes in Section 2.17. We have in addition the condition that the rope passes through (0,0).

If in actual flight $K$ and the coordinates of $A$ are measured, the shape of the rope is determined by (10.7). Having an equation for the curve of the rope we can substitute into (10.6), and find the tension force $T$ at each section of the rope. Knowing the area of the cross section of the rope we can compute the stress perpendicular to the section, and determine whether or not the rope is strong enough for its purpose.

Equation (10.7) came from the equilibrium of forces, and not from the stress-strain relations. We assumed nonrigidity of the rope, and an absence of flow of the rope material for the state considered. We can, of course, determine the strains at the points on the rope from Hooke's law if the stress is known at each point (the stress is $T$ divided by the cross sectional area).

Let us now consider the more general case where the rope may be moving with respect to the airplane. We shall again, for simplicity, treat states of the rope in the same vertical plane. We also neglect the wind forces acting directly on the rope. In Figure 170(a) we have pictured a section of the rope of length $\Delta s$ running from the point $P(x, y)$ to the point $Q(x + \Delta x, y + \Delta y)$. We suppose that with each point of the rope in the initial state we associate an arc length coordinate $s$ as before, so that $x$ and $y$ are functions of $s$, but not of time $t$. In Figure 170(b) we have drawn the section of the rope in a later state, where $Q'$ and $P'$ are derived from $Q$ and $P$ respectively. The $x$- and $y$-displacements $\alpha$ and $\beta$ of $P$ are functions of both $s$ and $t$. Assuming that the curve of Figure 170(b) is rectifiable, we can approximate the length of the arc $P'Q'$ by the length $c$ of the chord joining $P'$ and $Q'$. The relative elongation of the arc $PQ$ is then approximately $(c - PQ)/PQ$, that is,



(a) Initial state (t=0) (b) General state

Fig. 170. States of the glider rope.

$$\frac{\sqrt{(\Delta x + \Delta\alpha)^2 + (\Delta y + \Delta\beta)^2} - \Delta s}{\Delta s}. \tag{10.8}$$

The $\Delta$'s are due to $\Delta s$. Taking the limit of (10.8) as $\Delta s$ goes to zero we obtain the strain $\epsilon$ at $P'$, where

$$\epsilon = \sqrt{\left(\frac{dx}{ds} + \frac{\partial\alpha}{\partial s}\right)^2 + \left(\frac{dy}{ds} + \frac{\partial\beta}{\partial s}\right)^2} - 1.$$

The strain $\epsilon$ is in the direction of the tangent to the curve at $P'$, which tangent line is assumed to exist. By Hooke's law $\epsilon = kT$ for a constant $k$ and the tension force $T$ at $P'$, whence

$$\sqrt{\left(\frac{dx}{ds} + \frac{\partial\alpha}{\partial s}\right)^2 + \left(\frac{dy}{ds} + \frac{\partial\beta}{\partial s}\right)^2} = 1 + kT. \tag{10.9}$$

It is assumed that $x$ and $y$ are given functions of $s$ so that (10.9) involves the unknown functions $\alpha$, $\beta$, and $T$ of $s$ and $t$. We suppose that the rope density is uniform. The $x$-component of the inertial force on the element $P'Q'$ is minus the quantity

$$\frac{w}{g}\,\Delta s\,\frac{\partial^2(x + \alpha + \zeta\Delta x + \eta\Delta\alpha)}{\partial t^2}$$

for the weight $w$ per unit length, the magnitude $g$ of the acceleration of gravity, and a point along $P'Q'$ for which $\eta$ and $\zeta$ have the proper values. Since, however, $x$, $\zeta$, and $\Delta x$ are independent of $t$ this component becomes

$$\frac{w}{g}\,\Delta s\,\frac{\partial^2(\alpha + \eta\Delta\alpha)}{\partial t^2}.$$

For dynamic equilibrium we must have this term on the right of equation (10.4) in which $x$ is to be replaced by $(x + \alpha)$. Dividing both sides of this relation by $\Delta s$ and taking the limit as $\Delta s$ goes to zero (assuming that the limits exist) we find that

$$\frac{\partial\left[T\dfrac{\partial(x + \alpha)}{\partial s}\right]}{\partial s} = \frac{w}{g}\frac{\partial\alpha^2}{\partial t^2}. \tag{10.10}$$

Similarly, we obtain

$$\frac{\partial\left[T\dfrac{\partial(y + \beta)}{\partial s}\right]}{\partial s} = \frac{w}{g}\frac{\partial^2\beta}{\partial t^2} + w. \tag{10.11}$$

Equations (10.9), (10.10), and (10.11) are the three desired equations in the functions $\alpha$, $\beta$, and $T$. To determine the solution we also need the differential equations of the glider, and initial, as well as boundary conditions. The boundary condition is simply that $\alpha = \beta = 0$ at the point $(0, 0)$.

The reader will note that we did not use the notations $u$ and $v$ instead of $\alpha$ and $\beta$. The reason for this is that $\alpha$ and $\beta$ are not necessarily the displacements in the $x$- and $y$-directions of $P$ due to stresses in the rope. In general $\alpha$ and $\beta$ involve displacements due to motion of the rope as a rigid body, as well as displacements because of the tension in the rope. These displacements are additive. By the use of $\epsilon$ above to obtain (10.9) we were essentially able to extract from $\alpha$ and $\beta$ the displacements due to the stresses.

Consider now the action of the wind on the rope. Equations (10.10) and (10.11) must now be modified by adding corresponding terms to the right of these equations. These terms will be functions of the angle of inclination $\theta$ at $P'$, the speed $v$ of the wind due to the motion of the airplane, and the $x$- and $y$-components $\partial x/\partial t$, $\partial y/\partial t$ of the velocity of the point $P'$. Suppose that $\partial x/\partial t$ and $\partial y/\partial t$ are numerically small compared to $v$. The $x$-component of force due to $v$ will yield a term on the right of

(10.10) that increases with $|\theta|$ and $v$. From Part V it will appear reasonable to take this term to be proportional to $|\theta|\cdot v^2$. There are similar expressions for the other wind terms.

Equation (10.9) is too complicated to use in practice. It is reasonable to suppose that $\partial\alpha/\partial s$ is small compared with $dx/ds$, and that $dx/ds$ is always near 1, so that

$$\frac{dx}{ds} + \frac{\partial\alpha}{\partial s}$$

in (10.9) can be replaced by 1. Further $dy/ds$ may be small compared to to $\partial\beta/\partial s$, so that (10.9) can be replaced by

$$\sqrt{1 + \left(\frac{\partial\beta}{\partial s}\right)^2} = 1 + kT.$$

If we also assume that $\partial\beta/\partial s$ remains reasonably small we have

$$\frac{1}{2}\left(\frac{\partial\beta}{\partial s}\right)^2 = kT$$

by the binomial expansion. This relation is still nonlinear but is much better than (10.9).

**10.3. Thin beams.** We shall treat the case of a beam fixed at one end as shown in Figure 171, and supporting a load $W$ at the other end.

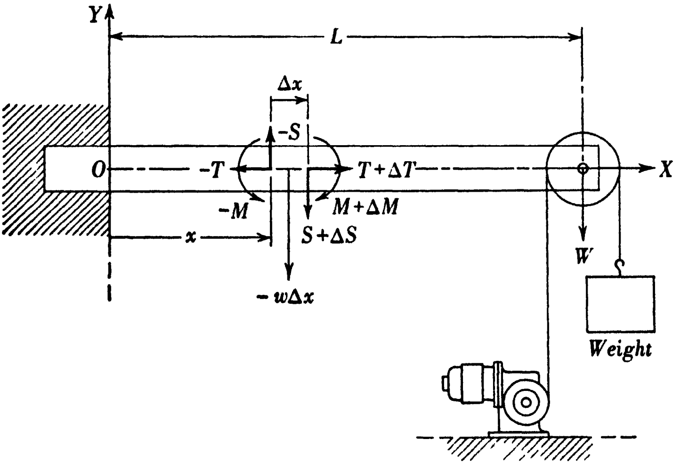


Fig. 171. Thin beam.

As an example, the weight $W$ might be caused by use of the beam for a hoist as shown. We suppose that the beam has constant cross section, as is ordinarily true in practice. We take the $x$-axis along the axis of the beam passing through the centroids of perpendicular cross sections; we take the

$y$-axis vertical, and the origin at the left end which is fixed. We assume that the beam is thin, that is, the dimension of the beam perpendicular to the $xy$-plane is not large compared with the other beam dimensions, a condition generally satisfied. We can then apply the two-dimensional treatment of Figure 171. Since in practice the amount of bending of a beam under load is small we have pictured the beam in Figure 171 to be horizontal. Actually the right end is lower than the left end. We suppose that the beam is of constant density, weighing $w$ units per unit of length in the $x$-direction. We let $L$ be the length of the beam.

Normally the *weight* of the beam can be *neglected* in comparison with the loads on the beam. We shall, however, include it to show how we treat a uniform vertical loading of the beam.

We assume the existence of the derivatives to be used in the argument.

We cut the beam at a section $A$ where the points of $A$ have the coordinate $x$, but keep the force distribution acting on the beam to the left of $A$ the same. In what follows *section* will refer to a cross section perpendicular to the axis of the beam. The part of the beam to the right of $A$ applies stresses to the part to the left of $A$. These stresses yield an $x$-component of force represented by $T$ and a $y$-component of force represented by $S$. The horizontal stresses also yield a moment $M$ about a line $l$ through the $x$-axis at the origin $O$ perpendicular to the $xy$-plane. Thus the part of the beam to the left of $A$ applies forces $-T$, $-S$, and a moment $-M$ to the part to the right of $A$. We cut the beam at another section $B$ where the points have the coordinate $(x + \Delta x)$. The part of the beam to the right of $B$ applies stresses on the part to the left, which stresses yield forces represented by $(T + \Delta T)$, $(S + \Delta S)$, and moment $(M + \Delta M)$ corresponding to $T$, $S$, and $M$ for the section $A$. Since the beam segment $AB$ is in equilibrium, the horizontal and vertical components of force on the segment vanish, whence

$$\Delta T = 0, \qquad \Delta S = w\Delta x. \tag{10.12}$$

Dividing by $\Delta x$ and taking the limit as $\Delta x$ goes to zero we have

$$\frac{dT}{dx} = 0, \qquad \frac{dS}{dx} = w. \tag{10.13}$$

The equations in (10.13) are valid throughout the beam from $x = 0$ to $x = L$. We shall assume that the weight $W$ acts at the very end of the beam, that is, at $x = L$.

The total moment of the forces on the beam segment $AB$ with respect to $l$ vanishes, whence (with the proper sign convention)

$$\Delta M + x\Delta S + (S + \Delta S)\Delta x = \left(x + \frac{\Delta x}{2}\right)(w\Delta x). \tag{10.14}$$

Dividing by $\Delta x$, taking the limit as $\Delta x$ goes to zero, and making use of the second equation in (10.13) we obtain

$$\frac{dM}{dx} = -S. \tag{10.15}$$

Thus we have derived three differential equations in $T$, $S$, and $M$. An examination of the method of solution reveals that the following boundary conditions are sufficient:

$$x = L, \qquad S = -W, \qquad M = -LW.$$

By (10.13) the tension $T$ is a constant, and since there is no tension at the right end of the beam the tension vanishes. In practice the tension $T$ would not vanish along the beam, although it would generally be negligible. We obtained the vanishing of $T$ because we neglected the bending of the beam.

The moment $M$ is generally referred to as the *bending moment*.

The deflection of the beam is often desired. To derive it we must make use of the elastic properties of the material. Thus far we managed without the stress-strain relations, although we have made use of the nonrigidity of the beam in treating internal forces. Since we would expect more bending for a greater bending moment $M$, it is reasonable to assume that $M$ is proportional to the curvature, or what is the same thing, is proportional to the reciprocal of the radius $R$ of curvature, so that

$$M = \frac{k}{R} \tag{10.16}$$

for a constant $k$. In Section 10.8 mathematical reasons for relation (10.16) will be given. This relation is known as the *Euler-Bernoulli law*. The constant $k$ is $EI$ where $E$ is Young's modulus and $I$ is the moment of inertia of a cross section of the beam with respect to $l$ (or a line parallel to $l$ through the centroid of the section). If $dy/dx$ is numerically small compared to 1 the curvature is approximately equal to $d^2y/dx^2$, whence (10.16) becomes

$$M = k\frac{d^2y}{dx^2}. \tag{10.17}$$

By (10.17) and the previously derived relations of this section we can solve for an equation of the curve of the beam, or more precisely the *axis* of the beam.

Suppose now that the beam is in motion. We assume that the density can be considered constant. With $g$ denoting the magnitude of the acceleration of gravity we have the $y$-component $w(\partial^2y/\partial t^2)\Delta x/g$, which must be added to the right of the second equation given in (10.12). The derivative in this component is evaluated at some properly chosen point on the section $AB$. We can also incorporate on the right of (10.14) the moment $w(x + \zeta\Delta x)(\partial^2y/\partial t^2)\Delta x/g$, where $\zeta$ is a number in the range 0 to 1. Since there is relatively little movement of the parts of the beam in the $x$-direction we can neglect this. Where the beam is used for a hoist as in Figure 171, the force $W$ at the end of the beam would probably vary, so that the corresponding end condition should be changed.

In Figure 172 we have pictured a *couple*, which is a pair of equal but oppositely directed forces in the same plane. We let $F$ designate the magnitude of these forces, and let $d$ be the distance between the lines of these forces. The reader can easily verify that the magnitude of the moment of these forces is the product $Fd$ with respect to any line perpendicular to the plane of these forces. The bending moment $M$ above is often considered as produced by a couple for which the magnitude of $M$ is $Fd$.

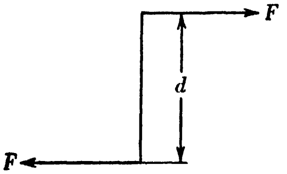


Fig. 172. Couple.

In this section we have treated a beam with one end fixed and the other end free. The *fixed end*, often called *clamped* or *built-in* is mathematically characterized by the fact that $y$ and $dy/dx$ are given (for example, at the centroid of this end). At a free end we have $d^2y/dx^2 = 0$ in view of (10.17). By (10.15) we have $d^3y/dx^3 = 0$ if there is no load at this end.

It is convenient in practice to use other end conditions. Thus at a hinged end, the beam end is geometrically fixed (at least at one point), but is free to rotate about this end. In this event we have the boundary condition $d^2y/dx^2 = 0$ from (10.17) and the given value of $y$ for a point on this end. We can also have an elastically clamped end with $y$ given and

$$\frac{d^2y}{dx^2} - k_1\frac{dy}{dx} = 0$$

for some real number $k_1$. Other boundary conditions for beam ends are possible.

When we pass a concentrated load $P$ the shear force changes by an amount equal to $P$.

### 10.4. Buckling of a beam.

Let us treat a vertical beam as in Figure 173, which is part of a structure, so that for practical purposes we may consider the top end $C$ as able to move in the vertical direction only. Even though the end $C$ moves horizontally, as long as its movement is not too large, we may imagine that the $x$-axis moves with it, where the $x$-axis passes through $C$ and the bottom end $O$ of the beam. The bottom end $O$ of the beam is assumed to be hinged. We suppose that the top end $C$

is loaded with a fixed force $L$ acting downward. We know from experience that if $L$ has the proper value the beam will buckle.

We apply methods of Section 10.3 except that the beam is vertical instead of horizontal. Since values of $L$ in which we are normally interested are usually large compared to the weight of the beam, we shall neglect the weight of the beam. The column generally remains nearly straight and therefore it appears reasonable to neglect shear. From the first equation of (10.13) and the boundary condition $T = -L$ at $C$ it follows that

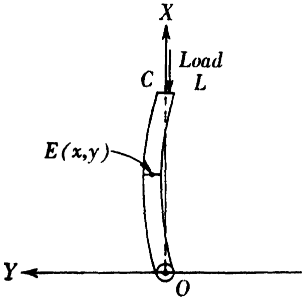


Fig. 173. Buckling beam.

$T = -L$ along the beam. Here the force $T$ is one of compression. Let the point $(x, y)$ be the centroid of a cross section $E$ of the beam. The bending moment $M$ (moment with respect to a line through $O$ perpendicular to the $xy$-plane) at $E$ satisfies the equation

$$M = -Ly. \tag{10.18}$$

By (10.17) and (10.18) we have (assuming that $d^2y/dx^2$ exists)

$$k\frac{d^2y}{dx^2} + Ly = 0 \tag{10.19}$$

The boundary conditions $x = 0$, $y = 0$; $x = L$, $y = 0$ apply.

The techniques used in Sections 10.3 and 10.4 for single beams apply to structures composed of a network of several beams.

If a beam is not *thin*, but is still of constant cross section, the above methods generally apply as long as the length of the beam is much more than the maximum chord of a perpendicular cross section.

**10.5. A uniformly loaded continuous beam.** In treating horizontal beams in structures it is often convenient to imagine that they are uniformly loaded. Such a beam may rest on several supports, often columns, and is called a *continuous beam.* We shall treat the example where one end $O$ of a beam $OB$ is hinged as in Figure 174, while the other end $B$ rests on a support, and the beam is further supported at a point $A$

between $O$ and $B$. We let $L$ designate the distance from $O$ to $A$, and $N$ the distance from $O$ to $B$. We let $w$ be the load per unit length along the beam.

By (10.13), (10.15), and (10.17) we have the differential equation (assuming that $d^4y/dx^4$ exists)

$$\frac{d^4y}{dx^4} = -\frac{w}{k}. \tag{10.20}$$

The general solution of (10.20) involves four arbitrary constants. We can think of the beam as made up of two parts, $OA$ and $AB$. The differential equation (10.20) thus applies to each part. However, since there are two parts there are eight arbitrary constants in the solutions of (10.20) for the beam. These constants can be determined from the conditions

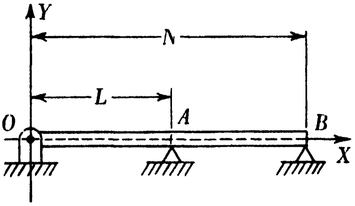


Fig. 174. Continuous beam.

$$x = 0,\quad y = 0; \qquad x = L,\quad y = 0; \qquad x = N,\quad y = 0$$

giving the altitudes of the supports, the conditions

$$x = 0,\quad \frac{d^2y}{dx^2} = 0; \qquad x = N,\quad \frac{d^2y}{dx^2} = 0,$$

which state that there are no bending moments at the supports $O$ and $B$, and the conditions that $dy/dx$ and $d^2y/dx^2$ are continuous at the support $A$; that is, the values of these derivatives at the point $A$ as given by the solution of (10.20) for the section $OA$ are equal to the values of the corresponding derivatives from the solution for the beam section $AB$. Since the condition $x = L$, $y = 0$ applies to both solutions we have the required eight conditions on the eight arbitrary constants.

**10.6. Two-beam hoist.** We shall treat a hoist made from two beams as in Figure 175. The horizontal beam $AC$ of length $N$ is attached at $A$ to a vertical supporting member that is rigid for practical purposes. At a point $B$ a distance $L$ from $A$ the horizontal beam $AC$ joins with a slant beam $DB$. The slant beam $DB$ is attached to the rigid supporting member at $D$. We shall suppose that for practical purposes all joints are hinged. At the free end $C$ of the beam $AC$ there is a pulley (not shown in the figure) supporting a weight $W$ through a rope or cable.

Since for practical purposes the weight $W$ is normally large compared to that of the beams, we shall neglect the mass of the beams. We let $\alpha$

designate the angle the slanted member makes with the horizontal, and $F$ the magnitude of the reaction of the hinge at $D$ on the member $DB$. There must then be a reaction of the beam $AC$ on the member $DB$ of the same magnitude but oppositely directed in order to have a force balance on the beam $DB$. These forces act along the same line $DB$ as is seen from taking the moment of the forces with respect to the point $D$. We can now apply the technique of Section 10.3 to the beam $DB$.

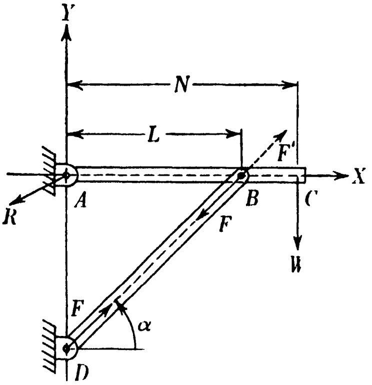


**Fig. 175. Hoist made from two beams.**

We introduce a horizontal $x$-axis and a vertical $y$-axis through $A$ as shown in the figure.

We restrict our attention to the horizontal member $AC$. This beam is subject to external forces composed of a force of magnitude $W$ acting downward at $C$, a force $F'$ of magnitude $F$ acting at $B$ in the direction of the angle $\alpha$ with the $x$-axis, and a reaction $R$ at $A$. We let $R_x$ and $R_y$ be the magnitudes of the $x$- and $y$-components of the reaction $R$.

The vanishing of the $x$- and $y$-components of the applied forces and the vanishing of the moment with respect to $A$ of the forces on the beam yield

$$\begin{aligned} R_y + F \sin \alpha &= W, \\ R_x + F \cos \alpha &= 0, \\ FL \sin \alpha &= NW. \end{aligned} \tag{10.21}$$

Equations (10.21) determine $F$, $R_x$, and $R_y$.

We use the notations $T$, $S$, and $M$ as in Section 10.3. We first treat the section $AB$ of the horizontal beam. By the first equation of (10.13) we have $T$ everywhere constant, and therefore equal to $F \cos \alpha$. We may suppose that the beam section $AB$ includes enough of the beam so that we may consider that the beam $DB$ acts on $AB$ only, whereas the stresses at the left end of the beam section $BC$ are due to the action of the beam section $AB$ on $BC$. In Figure 176 we have pictured the beam section $AB$

subject to the forces $R$, $T$, $S$, and moment $M$. By (10.21) and the equilibrium of the beam section $AB$ we have

$$S = F \sin \alpha - W \tag{10.22}$$

By (10.15) and the condition $x = 0$, $M = 0$ we have $M$ determined for the beam section $AB$. By (10.22) and the conditions $x = 0, y = 0; x = L, y = 0$ the deflection of $AB$ is determined.

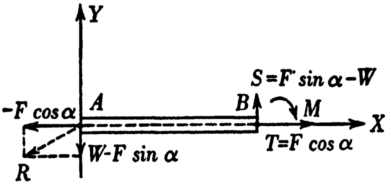


**Fig. 176. Beam section of Figure 175.**

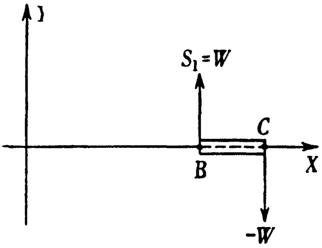


**Fig. 177. Beam section of Figure 175.**

We consider now the right end $BC$ of the horizontal beam, as pictured in Figure 177. From force equilibrium it follows that the shear force $S_1$ on $BC$ at $B$ is $W$. Thus in crossing $B$ we have a *discontinuity in the shear force* that changes from $S$ of (10.22) to $-W$. The shear force $S$ is $-W$ for any cross section of $BC$ perpendicular to the beam axis. Since there is no tension at the right end of $BC$ we have no tension at the left end. We thus also have a discontinuity in the tension as we cross $B$. By (10.13) the tension is zero along $BC$. By (10.15) and the condition that $M = 0$ at the free end $C$, the couple $M$ is determined for each cross section perpendicular to the beam axis. From (10.17), the condition that $x = L$, $y = 0$, and the continuity of the slope of the beam $AC$ at $B$, the deflection curve for the section $BC$ is obtained.

The analysis of the present section illustrates how a *problem involving discontinuities,* such as the shear force at $B$, *can be treated by solving for the continuous sections,* and *then piecing the solutions* for these sections *together* by making use of conditions which must hold at the junctions.

**10.7. Differential equations and boundary conditions for two-dimensional problems.** In Sections 10.1–10.6 we did not use the technique of treating stresses and strains at a general point in the body. Because of the difficulty of treating the partial differential equations involved, it is advisable to avoid the point treatment where possible. However, even when the point treatment is not used, the general theory of elasticity serves as a valuable guide for finding simplifying assumptions. From the previous sections it is clear that only dominant forces should be retained, and forces on entire sections should be used in preference to stresses at points. Where considerable precision is desired, or we wish to determine mathematically the validity of simplifying assumptions, it may be desirable to have recourse to the point method.

A good many problems in elasticity can be reduced to the study of stresses and strains in one plane. This is true of a thin plane plate, which we idealize to a region of a plane, if the externally applied stresses on the plate occur at the boundary of the plate in the plane of the plate, and are distributed uniformly over the thickness of the plate. Another common example is the case of a long thin horizontal beam of constant cross section uniformly loaded. If we analyze the stresses far enough away from the ends the deformation is essentially plane.

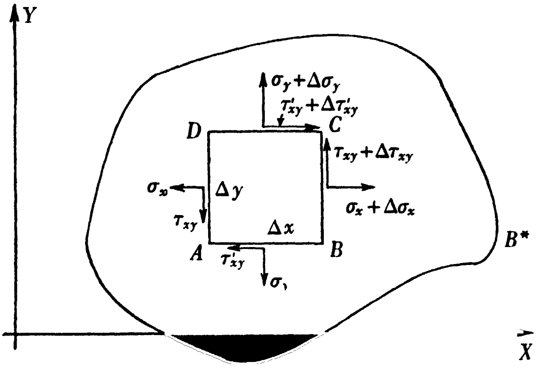

Y
$\sigma_y+\Delta\sigma_y$
$\tau'_{xy}+\Delta\tau'_{xy}$
D
C
$\tau_{xy}+\Delta\tau_{xy}$
$\sigma_x$
$\Delta y$
$\sigma_x+\Delta\sigma_x$
$\tau_{xy}$
$\Delta x$
$B^*$
A
B
$\tau'_{xy}$
$\sigma_y$
X

Fig. 178. Element of a two-dimensional body.

We suppose now that we have a case of plane deformation with $\gamma_{yz} = \gamma_{zx} = \epsilon_z = 0$. It follows from (9.31) that $\tau_{zx} = \tau_{yz} = 0$. However

$$\sigma_z = \mu(\sigma_x + \sigma_y).$$

The stress $\sigma_z$ maintains the plane deformation.

By (9.10) we have the equations

$$\epsilon_x = \frac{\partial u}{\partial x}, \; \epsilon_y = \frac{\partial v}{\partial y}, \; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{10.23}$$

By (9.13) the following compatibility equation is valid

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{10.24}$$

By (9.31), neglecting terms in $\mu^2$, we have the stress-strain relations

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E}, \; \epsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_x}{E}, \; \gamma_{xy} = \frac{\tau_{xy}}{G}. \tag{10.25}$$

Equations (10.25) also apply for the case of plane stress, where $\sigma_z = \tau_{yz} = \tau_{zx} = 0$. To complete the equations for the two-dimensional case we must have those expressing equilibrium.

In Figure 178 we have a two-dimensional body $B^*$, and a rectangular element $ABCD$ with sides parallel to the $x$- and $y$-axes. The dimensions of the element are $\Delta x$ and $\Delta y$ respectively. We may take the stress to be force density with respect to length (force per unit length), rather than area. On each side of the element we have indicated average stresses, so that, for example $\sigma_x \Delta y$ represents the normal force on the side $AD$ of the element, $(\sigma_x + \Delta\sigma_x)\Delta y$ the normal force on the side $BC$, and $\tau'_{xy}\Delta x$ the tangential force on the side $AB$, and finally $(\tau'_{xy} + \Delta\tau'_{xy})$ the tangential force on the side $DC$. We let $F_x \Delta x \Delta y$ and $F_y \Delta x \Delta y$ represent the body forces on the element in the $x$- and $y$-directions. From the vanishing of the forces in the $x$-direction we have

$$(\sigma_x + \Delta\sigma_x)\Delta y - \sigma_x \Delta y + (\tau'_{xy} + \Delta\tau'_{xy})\Delta x - \tau'_{xy}\Delta x + F_x \Delta x \Delta y = 0. \tag{10.26}$$

For the $y$-direction we have a corresponding formula. Dividing both sides of (10.26) by $\Delta x \Delta y$, taking the limit as the maximum of $\Delta x$, $\Delta y$ goes to zero (assuming that the limits exist), and setting $\tau_{xy} = \tau'_{xy}$ in the limit, we obtain

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0. \tag{10.27}$$

Similarly,

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0. \tag{10.28}$$

Equations (10.23), (10.24), (10.25), (10.27) and (10.28) give six equations in the five functions $\sigma_x$, $\sigma_y$, $\tau_{xy}$, $u$, and $v$. We note that the compatibility equation (10.24) can be expressed as a condition on the stresses.

We shall hereafter refer to the two-dimensional body as a *plate*. To determine the solution we must have equilibrium at the boundary between the external forces acting on the boundary and the internal stresses of the plate.

The stresses at the boundary of a body must be balanced by the surface forces. Let $X$ and $Y$ represent the $x$- and $y$-components of the surface force intensities (actually, intensities per unit length) at a point $P$ of the given body $B^*$. By (9.19) and (9.20) we have

$$\begin{aligned} X &= l\sigma_x + m\tau_{xy}, \\ Y &= l\tau_{xy} + m\sigma_y, \end{aligned} \tag{10.29}$$

in which $l$ and $m$ are direction cosines of the normal $N$ to the body $B^*$ at $P$, the normal in this two-dimensional case being assumed to lie in the $xy$-plane.

The equations (10.23), (10.24), (10.25), (10.27), (10.28), and (10.29) are the *fundamental equations* of elasticity for two dimensions.

**10.8. Beam bent by couples.** We consider a beam of constant cross section deformed by couples corresponding to bending moments $M$ and $-M$ as shown in Figure 179. The $x$-axis (before deformation) is taken along the beam through the line of centroids of cross sections of the beam, the $y$-axis is pointed downward at the left end $O$ of the beam, and the $z$-axis is horizontal.

The stresses contributing to a couple at one end of the beam are distributed along this end. Perhaps the simplest such distribution is that of Figure 180 where the stress is directly proportional to $y$. We write this stress as $\sigma_x$, where

$$\sigma_x = -\frac{Ey}{R} \tag{10.30}$$

for the modulus of elasticity $E$, and a constant $R$. We shall assume that there are no body forces, and no lateral forces on the beam.

Because of the *difficulty of solving the fundamental equations of elasticity directly, it is customary to guess at the solution* of these equations, or guess at part of the solution (as in Saint-Venant's semi-inverse method). Perhaps the simplest trial we can make in this case is to assume that the only stresses in the beam are the stresses given by (10.30).

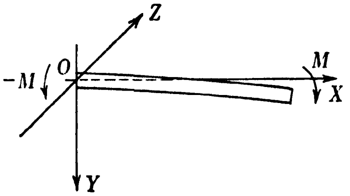


Fig. 179. Beam bent by couples.

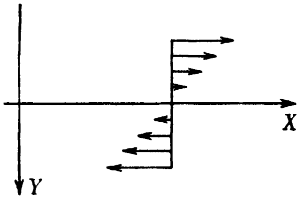


Fig. 180. Stress distribution on an end of a beam bent by couples.

From physical experience and the fact that the stresses act in the $x$-direction only, and do not vary in the $z$-direction, it seems reasonable to suppose that the deformation is plane, or approximately so. We therefore treat deformation in the $xy$-plane only.

Since the stresses are parallel to the $x$-axis, there can be no lateral forces in the $y$-direction in view of (10.29). This agrees with one of our assumptions. By (10.27) and (10.28) there are no body forces. By the stress-strain relations (10.25) we have

$$\epsilon_x = \frac{-y}{R}, \qquad \epsilon_y = \frac{\mu y}{R}, \qquad \gamma_{xy} = 0.$$

The compatibility equation (10.24) is satisfied. From the first two relations between the strains and the displacements given in (10.23) we have

$$u = -\frac{yx}{R} + f(y), \qquad v = \frac{\mu y^2}{2R} + g(x)$$

for functions $f(y)$ and $g(x)$. By the last relation of (10.23) we obtain

$$-\frac{x}{R} + f'(y) + g'(x) = 0. \tag{10.31}$$

By (10.31) we have $f'(y)$ a function of $x$, which can happen only if $f'(y)$ is a constant, whence

$$f(y) = Ay + B$$

for constants $A$ and $B$. Also,

$$g(x) = \frac{x^2}{2R} - Ax + C$$

for a constant $C$. The constants $B$ and $C$ represent a translatory motion of the beam. Without restriction on the generality of the method we may suppose the beam at the origin does not move, whence $B$ and $C$ vanish.

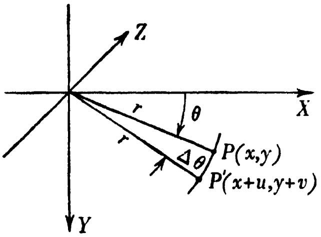


Fig. 181. Rotation of a point about the $z$-axis.

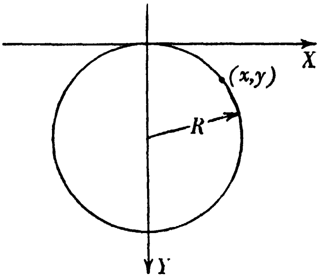


Fig. 182. Circle into which a beam is bent.

Consider now a point $P(x, y)$ of the beam as shown in Figure 181. In designating $P$ we leave out the $z$-coordinate. Suppose for the moment that the beam is rotated about the $z$-axis. We let $r$ and $\theta$ be the polar coordinates of $P$ in the $xy$-plane, so that after rotation the point $P$ has gone into a point $P'$ with polar coordinates $r$ and $(\theta + \Delta\theta)$ for the angle of rotation $\Delta\theta$. Since $x = r \cos \theta$, $y = r \sin \theta$, we have $dx = -r \sin \theta d\theta$, $dy = r \cos \theta d\theta$, whence the displacements $u$ and $v$ of $P$ can be approximated by

$$u = Ay, \qquad v = -Ax$$

for $A = -d\theta$.

Thus the terms $Ay$ and $-Ax$ in $f(y)$ and $g(x)$ correspond to rotation of the beam as a rigid body about the $z$-axis when the angle of rotation is small (so that we can write $dx$, $dy$ for $\Delta x$, $\Delta y$ respectively). Excluding such a rotation we drop the terms $Ay$, $-Ax$, and write

$$u = -\frac{yx}{R}, \qquad v = \frac{(x^2 + \mu y^2)}{2R}. \tag{10.32}$$

If $x$ is large compared to $y$, the displacement $u$ is small compared to $v$, and we can write

$$v = \frac{x^2}{2R}. \tag{10.33}$$

In Figure 182 we have pictured a circle of radius $R$ tangent to the $x$-axis with center at $x = 0$, $y = R$. From the equation $x^2 + (y - R)^2 = R^2$ for this circle we have

$$y = R\left[1 \pm \sqrt{1 - \frac{x^2}{R^2}}\right].$$

Expanding the radical by the binomial theorem and keeping the minus sign in front of the radical, we have

$$y = \frac{x^2}{2R} + \cdots$$

where the terms of higher degree in $x$ have been dropped. We are justified in dropping these terms if $R$ is large compared to $x$. We know from experience that a beam bent by end couples usually deforms very little even under heavy loads. It follows that the axis of the beam deforms for practical purposes into an arc of a circle of radius $R$.

For the stress distribution (10.30) the moment $M$ with respect to the $z$-axis is given by

$$M = \int_A\int \left(\frac{Ey}{R}\right) y\,dy\,dz$$

integrated over the region $A$ at the right end of the beam. It follows that

$$M = \frac{EI}{R}$$

for the moment of inertia $I$ of the end section, or for that matter any cross section perpendicular to the $x$-axis, with respect to the $z$-axis. From the formula for the radius of curvature we have $R$ approximately equal to $(d^2y/dx^2)^{-1}$ when $(dy/dx)^2$ is small compared to 1. Relation (10.17) is thus valid for $k = EI$.

It can be proved from uniqueness theory that our conjecture as to the internal stresses of the beam is the only one satisfying the applied force distribution.

**10.9. The principle of Saint-Venant.** In the previous section we showed how for the stress distribution of (10.30) we arrive at the Euler-Bernoulli relation (10.17). It is unlikely that we should have precisely this distribution. We therefore assume the following principle, which satisfies experience, and for which some mathematical justification can be made.

**Saint-Venant's principle.** *The stresses and strains in an elastic body at a large distance from a portion $\Delta S$ of the surface of the body, where large means relative to the linear dimensions of $\Delta S$, are approximately the same as the stresses and strains obtained when the force distribution on $\Delta S$*

*is replaced by another distribution yielding the same resultant force, and the same moment about some given line.*

For practical purposes a ratio of 10 to 1 between the distance from $\Delta S$ to the point of the body, where the stresses and strains are being studied, and the maximum linear dimension of $\Delta S$ is enough to insure that the force distribution on $\Delta S$ can be replaced by one equivalent in the sense of yielding the same resultant force and moment.

If we used any other stress distribution than (10.30) yielding the same couple, by Saint-Venant's principle we should get the same stresses and strains at a cross section sufficiently far removed from the ends of the beam as we obtained for the distribution (10.30). Unless the reader has evidence to the contrary he should accept the validity of the Euler-Bernoulli formula in each case he treats. Ordinarily only the force and moment are known, so that the analyst must assume a stress distribution to correspond to this force and moment.

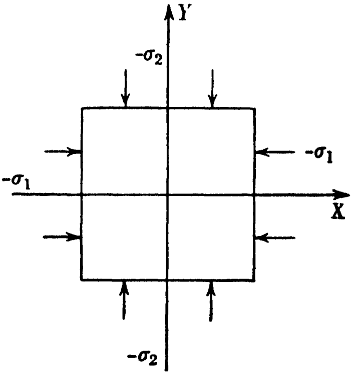


**Fig. 183. Plate under compression.**

**10.10. Rectangular plate under compression.** A rectangular plate in a machine is compressed by forces perpendicular to the four edges in such a way that the plate remains a rectangle. We picture the plate vertical as in Figure 183 with two edges horizontal. Since the plate is a rectangle it is reasonable to suppose that stresses are applied to the edges of the plate in a uniform manner. To obtain these stresses simply divide the total force on the edge by the area on which it acts. Let the vertical stress be denoted by $-\sigma_2$ as in Figure 183, and the horizontal stress by $-\sigma_1$, the minus signs denoting compression. We take the coordinate axes as shown with origin at the center of the plate and axes parallel to edges of the plate. Due to symmetry the center of the plate remains fixed, and points on the axes remain on their axes.

The conjecture that the stresses satisfy

$$\sigma_x = -\sigma_1, \quad \sigma_y = -\sigma_2$$

throughout the plate is perhaps the most obvious one. By (10.23), (10.25), and the assumption that the motion of the points on the plate is due to the deformation only, we obtain

$$u = \frac{1}{E}(\mu\sigma_2 - \sigma_1)x, \quad v = \frac{1}{E}(\mu\sigma_1 - \sigma_2)y.$$

The fundamental elasticity equations are now satisfied.

It is often a trivial matter to set up the differential equations and boundary conditions for a deformed body. This is generally of little value because of the difficulty of solving systems of partial differential equations. It is necessary to set up the problem in a mathematical form in such a way that a solution can be readily obtained. For this purpose the analyst must ordinarily replace the boundary conditions by others or introduce boundary conditions so that he can conjecture a solution of the problem, or the problem can be solved by direct methods. Until he has obtained the solution he may not be sure that he has set the problem up properly.

Fortunately, the boundary conditions that are mathematically simplest often suffice. Thus a uniform stress distribution, or one varying linearly with distance are mathematically simple, and often lead to good solutions. We illustrated the use of a uniform distribution in this section. In Section 10.8 we showed how a couple could be replaced by a linearly varying stress distribution.

For the beam of Section 10.8 the stress $\sigma_x$ is related to the given moment $M$ by the equation

$$\sigma_x = -\frac{My}{I}. \tag{10.34}$$

*If a couple with moment M acting on a plane or almost plane surface were given in some problem, we could set up coordinate axes as in Section 10.8 so that a stress distribution like (10.34) could be used on the given surface in place of the couple.*

*If both resultant forces and moments apply to a given body, we should employ the principle of superposition, using a stress distribution for the forces in addition to a stress distribution for the moments.*

**10.11. Dimensional analysis applied to a stretched string.** Consider a string of length $l$ with density $\rho$ (mass per unit length), under a constant tension $F$. We wish a formula for the fundamental frequency $f$ of vibration of the string. We take the primary quantities to be length $L$, mass $M$, and time $T$. The dimensions of the secondary quantities follow.

| QUANTITY | DIMENSIONS |
|---|---|
| $l$ | $L$ |
| $F$ | $MLT^{-2}$ |
| $\rho$ | $ML^{-1}$ |
| $f$ | $T^{-1}$ |

By the Buckingham pi theorem we may expect a dimensionless $\pi$-term. We take this term in the form

$$fl^aF^b\rho^c.$$

Since this term is dimensionless in $L$, $M$, and $T$ we have

$$\begin{aligned} a + b - c &= 0, \\ -2b &= 1, \\ b + c &= 0. \end{aligned}$$

This system of equations has the solution

$$b = -\tfrac{1}{2},\ c = \tfrac{1}{2},\ a = 1.$$

By the Buckingham pi theorem we have $flF^{-\frac{1}{2}}\rho^{\frac{1}{2}}$ equal to a constant $K$. It follows that

$$f = \frac{K}{l}\sqrt{\frac{F}{\rho}}.$$

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2. John Prescott, *Applied Elasticity*, Dover, New York, 1946.

CHAPTER

11

# *Elasticity in three dimensions*

**11.1. Equilibrium equations and boundary conditions for three dimensions.** In Section 9.7 we derived the stress-strain relations for three dimensions. To complete the mathematical picture we also need the equations of equilibrium of forces, and the boundary conditions. It is a simple matter to extend the results of Section 10.7 to three dimensions. In place of (10.27) and (10.28) we derive

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x &= 0, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y &= 0, \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z &= 0 \end{aligned} \tag{11.1}$$

from the equilibrium of forces. Here $F_x$, $F_y$, and $F_z$ represent the components of the body force intensity $F$. If $X$, $Y$, and $Z$ represent the $x$-, $y$-, and $z$-components of the surface force intensity at a point on the surface of the given body, the equations (10.29) generalize to

$$\begin{aligned} l\sigma_x + m\tau_{xy} + n\tau_{zx} &= X, \\ l\tau_{xy} + m\sigma_y + n\tau_{yz} &= Y, \\ l\tau_{zx} + m\tau_{yz} + n\sigma_z &= Z. \end{aligned} \tag{11.2}$$

There are many problems where one- and two-dimensional methods fail, and *these fundamental equations of three-dimensional elasticity* must be used.

**11.2. A rod stressed at the ends.** A rod in a machine is often under tension or compression on its ends. If the cross section of the rod perpendicular to some axis along the rod varies materially, we might treat the rod as one composed of several rods of uniform cross section joined end to end. We shall give a rigorous treatment of a rod of uniform cross section only, that is, a cylinder (not necessarily circular) subject only to a tension or compression force on its ends. We let $T$ be the magnitude of the force on an end of the rod, and let $A$ be the area of one end of this rod.

Since there is no bending moment $M$ on the ends we shall replace the given force by a uniform stress distribution, which in the case of tension will yield a stress $T/A$ to be designated by $\sigma$. We shall neglect gravitation, which is generally justified.

Let $L$ be the length of the rod, and let us set up a coordinate system as shown in Figure 184 with the $x$-axis parallel to a generator of the rod, and the left end on the $yz$-plane. We have $\tau_{xy} = \tau_{xz} = 0$, $\sigma_x = \sigma$ at $x = L$, and corresponding stresses on the end $x = 0$.

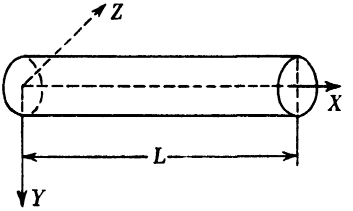


**Fig. 184. Body under end stresses.**

We assume that $\sigma_x = \sigma$ throughout the rod, and that the other stresses vanish. The equilibrium equations (11.1), the compatibility equations, and the boundary conditions (11.2) are now satisfied. The compatibility equations (9.13) are expressed in terms of strains, but can, of course, be written in terms of stresses. The equations (9.10) now become

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\sigma}{E}, & \frac{\partial v}{\partial y} &= -\frac{\mu\sigma}{E}, & \frac{\partial w}{\partial z} &= -\frac{\mu\sigma}{E}, \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= 0, & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} &= 0, & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} &= 0. \end{aligned} \tag{11.3}$$

which can easily be solved for the displacements $u$, $v$, and $w$. Since the fundamental elasticity equations are satisfied we know that our conjecture as to the internal stress distribution was correct.

To determine the solution of (11.3) for the displacements free of arbitrary constants it is necessary to use boundary conditions, such as that points on the $x$-axis remain on the $x$-axis.

**11.3. Three-dimensional treatment of a beam bent by couples.** Let us return to the example of Section 10.8 of a beam bent by couples. Assuming that the stress $\sigma_x$ of (10.30) is the only stress acting, the first three equations of (11.3) are replaced by

$$\frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = \frac{\mu y}{R}, \qquad \frac{\partial u}{\partial x} = -\frac{y}{R},$$

while the other three remain the same. It is a simple matter to verify that the equilibrium conditions, compatibility equations, and boundary conditions are satisfied. With no rigid body translation at the origin $O$ or rotation at $O$ about $O$ we have

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0$$

at $O$. Solution for the displacements yields

$$u = -\frac{yx}{R}, \qquad v = \frac{1}{2R}(x^2 + \mu y^2 - \mu z^2), \qquad w = \frac{\mu yz}{R}.$$

For $y$ and $z$ small compared to $x$, these displacements reduce to (10.33) as before, yielding again the Euler-Bernoulli formula (10.17).

**11.4. Torsion of a circular cylindrical shaft.** If a cylinder is twisted by couples acting at its ends, where these couples lie in the planes of the bases of the cylinder, the cylinder is said to be under *torsion.* Consider a right circular cylindrical shaft like the transmission of an automobile. In Figure 185 we have pictured such a shaft under torsion, with the $x$-axis along the center of the shaft, and one end of the shaft on the $yz$-plane. The other end of the shaft lies in a plane parallel to the $yz$-plane. Let $L$ be the length of the shaft.

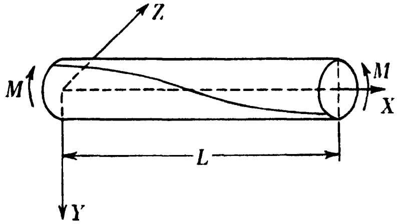


**Fig. 185. Circular shaft under torsion.**

We shall suppose that the stresses due to surface forces on the end of the shaft are symmetrically distributed about the axis of the shaft. Because of symmetry, the points of the untwisted shaft which lie on a circle with center on the $x$-axis, in a plane perpendicular to this axis, go into points on such a circle after the shaft is twisted. We shall venture the guess that such circles in the same plane go into circles in the same plane, that is, plane cross sections perpendicular to the $x$-axis go into plane cross sections perpendicular to this axis. Let the left end of the shaft be fixed relative to the coordinate system. We shall try the stronger assumption that the cross sections of the shaft perpendicular to the $x$-axis rotate about the $x$-axis. On the basis of experience we would expect such sections to rotate more when farther away from the fixed end. Let the angle of rotation be denoted by $\theta$. The simplest law connecting $\theta$ with the section is the following direct proportionality

$$\theta = ax \tag{11.4}$$

for a constant $a$. Here $a$ is the *angle of twist* per unit length.

We make the further assumption that points on the shaft do not displace in the $x$-direction, so that $u = 0$. By the treatment of rotations in Section 10.8 we have

$$v = -\theta z, \qquad w = \theta y,$$

whence

$$v = -axz, \qquad w = axy.$$

By (9.10) and (9.31) we have

$$\tau_{xz} = Gay, \qquad \tau_{xy} = -Gaz, \tag{11.5}$$

while the other stresses vanish. The constant $G$ is the modulus of rigidity introduced in (9.31). Here we prefer $\tau_{xz}$ to $\tau_{zx}$ because we are treating stresses perpendicular to the $x$-axis. With no body forces acting, the stresses are readily seen to satisfy the equations (11.1) of equilibrium, and the equations of compatibility (9.13). The last two boundary conditions in (11.2) for the lateral surface, with no lateral surface forces are seen at once to be satisfied, inasmuch as $l = 0$ for a normal to the surface of the shaft, and $\sigma_y$, $\sigma_z$, $\tau_{yz}$, $Y$, $Z$ vanish. Since $l$ and $X$ vanish, the first equation in (11.2) reduces to

$$m\tau_{xy} + n\tau_{xz} = 0.$$

If $R$ is the radius of the shaft and $(x, y, z)$ the point on the surface at which the boundary conditions are being studied, we have

$$m = \frac{y}{R}, \quad n = \frac{z}{R}$$

It follows from the formulas for the stresses $\tau_{xz}$, $\tau_{xy}$ that the first equation of (11.2) is satisfied.

It remains to consider the boundary conditions on the ends. The nonvanishing stresses on the ends are given by (11.5). Integration reveals that with respect to the $y$- and $z$-axes the moments corresponding to this stress distribution vanish. With respect to the $x$-axis we have the moment

$$M = GaI_0, \tag{11.6}$$

where $I_0$ is the moment of inertia of a cross section perpendicular to the $x$-axis with respect to the center of this section.

From uniqueness theorems we can show that the solution above is the correct one for the applied stresses.

By Saint-Venant's principle we may suppose that the solution above applies for any stress distribution on the ends yielding the same couple $M$, except near these ends.

In practice $G$, $I_0$, and $M$ are known, so that the quantity $a$ can be computed from (11.6). We then assume the stress distribution (11.5), whence the strains are as described above.

In the example of this section we made a conjecture as to the boundary conditions. It was necessary to carry through the solution to determine whether or not the assumptions were correct.

**11.5. Balloon under pressure.** Let us treat a spherical balloon under expansion due to the excess of the pressure of the gas inside of the balloon over the pressure outside. The treatment to follow applies equally well to a spherical tank with a thin wall and with a surface under a uniform pressure due to the presence of a liquid or gas within the tank. We let $p$ be the net pressure, that is, the pressure on the inside surface minus the pressure on the outside.

We introduce an element on the surface of the balloon as in Figure 186. Here the element is a spherical quadrilateral formed by four great circles, where the sides of the quadrilateral are equal, and the angles are equal.

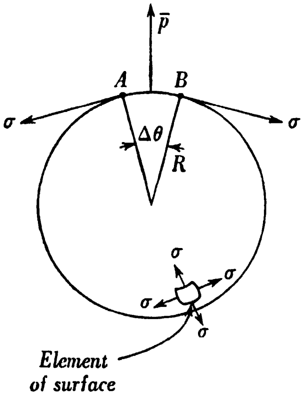


**Fig. 186. Balloon under pressure.**

Let $\Delta\theta$ be the radian measure of one side, and $R$ the radius of the balloon, so that the length of a side of the quadrilateral is $R\Delta\theta$. We let $AB$ designate such a side as in the figure. By reason of the symmetry of the balloon the resultant force on a side of the element, because of the action of the rest of the balloon, is normal to this side at the middle. If there were a component of this resultant force tangent to the side, this would imply that one direction along this arc (that of the tangent component of the force) was preferred to the other. Because of symmetry there can be no preference for one direction over the other, and for this reason we infer that there is no such tangential component. We write the magnitude of the resultant force as $\sigma R\Delta\theta$ for a stress represented by $\sigma$. The force due to pressure on the element has a resultant normal to the center $C$ of the element. We designate this force by $\bar{p}$ times the area of the element. This area is a function $f(\Delta\theta)$ of $\Delta\theta$, which goes to zero with $\Delta\theta$. The resultant force due to pressure is then $\bar{p}f(\Delta\theta)$. We take the component, along the normal to the element at $C$, of the tension force on one side of the element. This component is represented by $\sigma R\Delta\theta \sin \frac{\Delta\theta}{2}\cdot$ Since there are four sides of the element, and we can replace $\sin \frac{\Delta\theta}{2}$ by $\frac{\Delta\theta}{2}$ for small $\Delta\theta$, the total radial component due to tension is approximately $2\sigma R(\Delta\theta)^2$. Equating this to the resultant pressure force we have

$$\bar{p}f(\Delta\theta) = 2\sigma R(\Delta\theta)^2. \tag{11.7}$$

The quantity $f(\Delta\theta)$ is an area, whence it is not surprising that the quantity $f(\Delta\theta)/(\Delta\theta)^2$ has a limit as $\Delta\theta$ goes to zero while the sides and angles of the quadrilateral remain respectively equal to each other. Dividing both sides of (11.7) by $(\Delta\theta)^2$ and taking the limit as $\Delta\theta$ goes to zero we have

$$p \lim_{\Delta\theta \to 0} \frac{f(\Delta\theta)}{(\Delta\theta)^2} = 2\sigma R. \tag{11.8}$$

By spherical trigonometry $f(\Delta\theta)$ can be readily computed, and the limit in (11.8) found to be $R^2$. If $\Delta\theta$ is small the quadrilateral is approximately a square and the area is approximately the square of the length $R\Delta\theta$ of one side. The relation (11.8) gives the normal stress $\sigma$ where $\sigma = (pR)/2$, which is the same for each direction due to symmetry.

Treating a small region on the surface of the sphere as plane, we can apply two-dimensional theory and obtain the longitudinal strain. From the strain at each point on a great circle we can derive at once the circumferential or radial elongation of the sphere under the influence of the given pressure.

In this three-dimensional example we found it convenient to study an element of the sphere rather than use the elasticity equations derived for three dimensions. In various examples the direct attack is often to be preferred, especially when some dimension can be neglected, in this case the thickness of the balloon material.

**11.6. Bottom of a barrel in the form of a membrane.** Suppose that the bottom of a barrel is in the form of a taut circular membrane attached all around its edge to the barrel. This membrane is under uniform normal pressure from a liquid in the barrel if the barrel is vertical, which is assumed. We take the radius of the membrane to be $R$ and set up an $xy$-coordinate system with origin at the center of the membrane when it is undistorted, so that the perimeter of the membrane has the equation

$$x^2 + y^2 = R^2.$$

We introduce the $z$-axis so that it points downward as in Figure 187 where we have drawn a cross section of the membrane under pressure.

The deflection of the membrane is such that the $z$-coordinate of points on the membrane is small compared with the diameter of the membrane.

In Figure 188(a) we have indicated an element $ABDC$ of the membrane bounded by arcs $AB$ and $CD$ of circles with centers on the $z$-axis and in planes perpendicular to the $z$-axis. The sides $AC$ and $BD$ lie in planes through the $z$-axis. By reason of symmetry we can conclude that the resultant force acting on the edge $AB$ is normal to $AB$ at its center; the proof runs along lines such as the following. Suppose that this force had a component $R$ as in Figure 188(b) tangent to $AB$ at its center. As far as the action of the membrane on the element along the edge $AB$ is concerned the same geometric conditions prevail everywhere along this edge. We could therefore just as easily have $R$ pointed in the reverse direction.

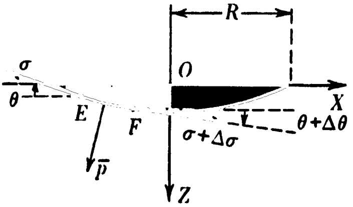
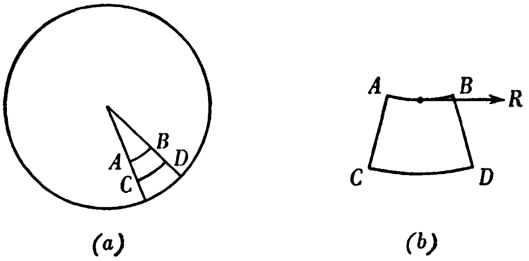


Fig. 187. Bottom of a barrel.



(a) (b)

Fig. 188. Membrane.

Because we expect the stresses and forces to be unique, we decide that a component $R$ is impossible. Similar considerations apply to the edges $AC$ and $BD$.

In going to the limit as the maximum linear dimension of the element $ABDC$ goes to zero we derive a radial stress, and a stress perpendicular to the radius. We shall make the simplest assumption we can about these stresses, namely that they are equal, in fact we shall suppose that there are no shear stresses, and that the longitudinal stress represented by $\sigma$ is the same in all directions. Symmetry considerations show that along the periphery of the membrane we have only a normal stress.

*A membrane cannot support bending moments, compression or shear stresses; in fact, it can support only tension.*

We consider the usual rectangular element of area $\Delta x \Delta y$ in the $xy$-plane. This is the projection on the $xy$-plane of an element of area on the distorted membrane of which two edges $E$ and $F$ are indicated in Figure 187. Since the membrane is practically plane we approximate the dimensions of the element on the surface by $\Delta x$ and $\Delta y$. Let $\theta$ be such that $\tan \theta = \partial z/\partial x$. The vertical component of the force on the edge $E$ is represented by $\sigma(\sin \theta)\Delta y$ for an angle $\theta$ evaluated at some properly chosen point on the edge $E$. For the edge $F$ we then have $(\sigma + \Delta\sigma)[\sin (\theta + \Delta\theta)]\Delta y$. The net vertical component corresponding to the stresses on the edges $E$ and $F$ is thus approximately

$$(\sigma + \Delta\sigma)(\theta + \Delta\theta)\Delta y - \sigma\theta\Delta y$$

inasmuch as $\theta$ and $(\theta + \Delta\theta)$ are small. This component is approximately $\Delta y[\sigma\Delta\theta + \theta\Delta\sigma]$. Similarly, from the other two edges of the element we have the vertical component $\Delta x[\sigma\Delta\alpha + \alpha\Delta\sigma]$ for an angle $\alpha$ defined by $\tan \alpha = \partial z/\partial y$, and average stress $\sigma$. The vertical component due to the pressure is $\bar{p}\Delta x\Delta y$ for a properly chosen vertical component $\bar{p}$ of the pressure $p$ on the element. The vanishing of the vertical component of the forces on the element yields the approximate relation

$$\bar{p}\Delta x\Delta y + \Delta y(\sigma\Delta\theta + \theta\Delta\sigma) + \Delta x(\sigma\Delta\alpha + \alpha\Delta\sigma) = 0. \qquad (11.9)$$

Dividing both sides of (11.9) by $\Delta x\Delta y$, and letting the maximum of $\Delta x$, $\Delta y$ go to zero we have (assuming that the limits exist)

$$p + \sigma\frac{\partial\theta}{\partial x} + \theta\frac{\partial\sigma}{\partial x} + \sigma\frac{\partial\alpha}{\partial y} + \alpha\frac{\partial\sigma}{\partial y} = 0.$$

Here we have used $p$ for the limit of $\bar{p}$. Since $\alpha$ and $\theta$ are small we drop the third and fifth terms to derive

$$p + \sigma\frac{\partial\theta}{\partial x} + \sigma\frac{\partial\alpha}{\partial y} = 0$$

The definitions of $\alpha$ and $\theta$ give

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{\sigma}. \qquad (11.10)$$

To equation (11.10) we must adjoin the boundary condition

$$x^2 + y^2 = R^2, \qquad z = 0.$$

In this example the membrane is understood to be so thin that for practical purposes it may be treated as a surface.

Because of the nature of the problem the above example can be more readily treated with polar coordinates.

**11.7. Energy method applied to a helical spring.** Energy considerations can often be used to advantage in treating elasticity problems.

In Section 11.4 we discussed torsion in cylindrical shafts. Consider one end of the shaft fixed. A twisting couple of moment $M$ rotated through an angle $\theta$ does the amount of work $W$ given by

$$W = M\theta \tag{11.11}$$

in view of (2.63). This is then the elastic energy stored in the shaft as a result of the work done on the shaft by the given couple.

In Figure 189 we have pictured a helical spring wound in the form of a right circular cylinder. Suppose that a resultant compression force of magnitude $F$ is applied to the ends of the spring. Where the wire (for the moment thought of as a curve) leaves the plane of one end of the spring there will be a twisting moment $M$ about the axis of the wire (not the axis of the spring), as well as other forces and moments which experiments show can be neglected. Assuming symmetry in the sense that the spring looks the same from both ends, there is an identical twisting moment $M$ at the other end of the spring about the axis of the wire. Experiment shows that the longitudinal deflection of the spring is due principally to torsion of the wire. The simplest assumption we can now make is that there is a constant couple $M$ at each perpendicular cross section $P$ of the wire tending to twist the wire. If $\theta$ is the total angle of rotation of the wire about its axis (the axis runs through the center of the wire), by (11.11) the energy stored in the wire is $M\theta$. In compressing the spring the force $F$ is applied to an end which moves a distance that we shall denote by $x$. The work $W$ done by the force $F$ is then

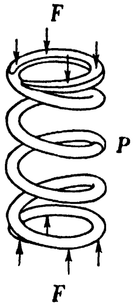


**Fig. 189. Helical spring.**

$$W = \int_0^x F dx. \tag{11.12}$$

Since the shape of the end of the spring does not change much with different uniform force distributions normal to the end of the spring in the direction of the spring axis, it follows that

$$M = KF$$

for a constant $K$. Let $L$ be the length of the wire. By (11.6) and $\theta = aL$

$$\theta = \frac{ML}{GI_0},$$

whence (11.11) becomes

$$W = \frac{M^2L}{GI_0} = \frac{K^2LF^2}{GI_0}.$$

In view of (11.12) we have now

$$\int_0^x F dx = \frac{K^2LF^2}{GI_0}. \tag{11.13}$$

We can differentiate (11.13) with respect to $x$, divide $F$ out of both sides of the resulting equation, and obtain a constant for $dF/dx$. Since $F = 0$ when $x = 0$, we can obtain a direct proportionality between $F$ and $x$.

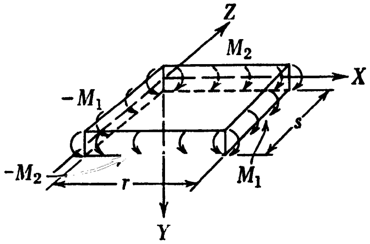


**Fig. 190. Bending of a rectangular plate by moments $\pm M_1$ and $\pm M_2$.**

The above analysis would have little value unless verified by experiment, inasmuch as we have by-passed the fundamental equations of elasticity because of the more or less complicated shape of the wire in the spring.

**11.8. Pure bending of plates.** A rectangular plate is a beam of constant rectangular cross section with one dimension small compared to the other two. We shall consider such a plate as pictured in Figure 190 where we apply bending moments $\pm M_1$ to the two edges parallel to the $z$-axis. The moment $M_1$ corresponds to the moment $M$ of Figure 179. We also apply moments $\pm M_2$ to the two edges parallel to the $x$-axis as shown.

Let us return to the beam of Figure 179 and the analysis of Section 11.3. We let $(x_1, y_1, z_1)$ be the coordinates of a point of the beam before deformation, and $(x, y, z)$ the coordinates of this point after deformation, whence

$$x = x_1 + u, \qquad y = y_1 + v, \qquad z = z_1 + w$$

for the displacements $u$, $v$, and $w$. Consider the cross section $x_1 = a$ of the beam before deformation. Since

$$x_1 = x - u, \qquad u = -\frac{y_1 x_1}{R}$$

the plane $x_1 = a$ goes into

$$x = a - \frac{ay}{R}, \tag{11.14}$$

where we have assumed that we could replace $y_1$ in $ay_1/R$ by $y$. Equation (11.14) is the equation of a plane, whence $x_1 = a$ deforms into a plane. This equation is satisfied by all points $(0, R, z)$ so that the planes (11.14) for the different values of $a$ intersect along the centers of curvature (see Section 10.8). In Figure 191 we have the cross section of the beam lying in the plane $z = z_1 = 0$. For each $a$ the section $x_1 = a$ goes into a line segment which when sufficiently extended passes through the point $(0, R)$.

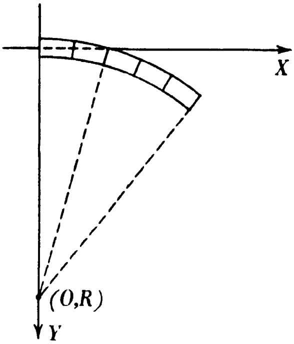


Fig. 191. Bent beam.

The lateral faces of the undeformed beam of Figure 179 are given by equations

$$y_1 = \pm b, \qquad z_1 = \pm c \tag{11.15}$$

for numbers $b$ and $c$. In the plane $x_1 = a$ the equations (11.15) define a rectangle.

Under the deformation the equations $z_1 = \pm c$ go into

$$z = \pm c\left(1 + \frac{\mu y}{R}\right), \tag{11.16}$$

where we have assumed that we could replace $y_1$ in $\mu y_1/R$ by $y$. The equations (11.16) are equations of planes inclined to the $xy$-plane. The sides $z_1 = \pm c$ in the plane $x_1 = a$ thus go into straight lines that are slanted instead of vertical.

Similarly, the equations $y_1 = \pm b$ go into

$$y = \frac{1}{2R}\left[x^2 + \mu(b^2 - z^2)\right] \pm b, \tag{11.17}$$

which are quadric surfaces. The straight lines $y_1 = \pm b$, $x_1 = a$ thus go approximately into the curves represented by

$$y = \frac{1}{2R}[a^2 + \mu(b^2 - z^2)] \pm b, \tag{11.18}$$

and (11.14). The equation (11.18) is a parabolic cylinder with elements parallel to the $x$-axis. The lines $y_1 = \pm b$, $x_1 = a$ thus go into parabolas, that except for translation lie on the surface

$$y = -\frac{\mu z^2}{2R}. \tag{11.19}$$

By Section 10.8 the equation (11.19) represents an arc of a circle for practical purposes if the deformation is not too large. It follows that the sides $y_1 = \pm b$, $x_1 = a$ go into arcs of circles of radius $R/\mu$, so that the rectangle defined by (11.15) goes into the shape indicated by the dashes in Figure

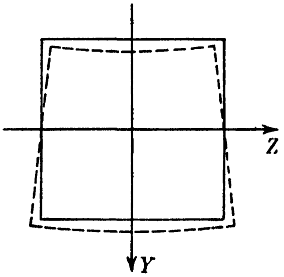


Fig. 192. Cross section of a beam.

192. Consider now the plate of Figure 190 with the moments $\pm M_1$ applied to the edges shown, but the moments $\pm M_2$ omitted. Let the dimensions of the plate in the $x$- and $z$-directions be denoted by $r$ and $s$ respectively. The moment of inertia $I_z$ of the edge $x = 0$ with respect to the $z$-axis is given by

$$I_z = \tfrac{2}{3}sh^3,$$

where the thickness of the plate is $2h$. By Section 10.8 we have

$$\frac{1}{R} = \frac{3M_1}{2Esh^3}$$

for the radius of curvature $R$. The quantity $1/R$ is the curvature of the surface curves of the plate in planes parallel to the $xy$-plane. For planes parallel to the $yz$-plane we have the corresponding curvature $-\mu/R$.

Suppose now that the moments $\pm M_1$ are applied to the edges of the plate parallel to the $z$-axis, and the moments $\pm M_2$ to the edges of the plate parallel to the $x$-axis as in Figure 190. Let $K_1$ and $K_2$ be the curvatures of the surface curves in planes parallel to the $xy$- and $yz$-planes, respectively. For practical purposes these curvatures are second derivatives. We assume that the principle of superposition applies so that

$$\begin{aligned} K_1 &= \frac{3}{2Eh^3}\left(\frac{M_1}{s} - \frac{\mu M_2}{r}\right), \\ K_2 &= \frac{3}{2Eh^3}\left(\frac{M_2}{r} - \frac{\mu M_1}{s}\right). \end{aligned} \tag{11.20}$$

It is assumed throughout that the deflections are small compared with the thickness of the plate.

Solving (11.20) we have

$$\begin{aligned} M_1 &= \frac{2sEh^3}{3(1-\mu^2)}(K_1 + \mu K_2), \\ M_2 &= \frac{2rEh^3}{3(1-\mu^2)}(K_2 + \mu K_1). \end{aligned}$$

For small deflections we have

$$K_1 = \frac{\partial^2 v}{\partial x^2}, \qquad K_2 = \frac{\partial^2 v}{\partial z^2}.$$

The *flexural rigidity* $D$ of the plate is defined by

$$D = \frac{2Eh^3}{3(1-\mu^2)}$$

so that

$$\begin{aligned} M_1 &= sD\left(\frac{\partial^2 v}{\partial x^2} + \mu\frac{\partial^2 v}{\partial z^2}\right), \\ M_2 &= rD\left(\frac{\partial^2 v}{\partial z^2} + \mu\frac{\partial^2 v}{\partial x^2}\right). \end{aligned} \tag{11.21}$$

By means of (11.21) the moments $M_1$ and $M_2$ can be computed from given properties of the deformed plate; or from given moments $M_1$ and $M_2$, the equations (11.21), and boundary conditions at the edges of the plate, we can obtain the deflections of the plate.

**11.9. Thermal stresses.** In the study of internal combustion engines, such as gas turbines, stresses in a body can be introduced by heating the body, such stresses being called *thermal stresses.* From experience we know that, in an isotropic body, heat causes identical longitudinal strains in all directions through the point being studied, provided the body is free to deform. If $T$ is the temperature that causes the thermal strains we have the strains $\epsilon_x = AT$, $\epsilon_y = AT$, $\epsilon_z = AT$ for a constant $A$. If the deformation is prevented we have corresponding thermal stresses. For steel a change of 12 degrees Fahrenheit in the temperature can cause a stress of 2,340 pounds per square inch.

**11.10. Principles of the analysis of elastic bodies.** In treating elastic bodies, we should if possible replace the body by one of simple geometric shape. Thus if we were making a rough treatment of the torsion of an elliptical rod, where the cross section of the rod does not deviate too much from a circle, we might replace this rod by one of the same material but circular cross section, where the radius of the circle is between the minor and major axes of the ellipse. Similarly, in an approximate analysis a hollow rod subject to bending might be replaced by a solid rod, provided the given rod is not too hollow. Where a rod has a varying cross section one might try treating it as a rod formed by joining two or more rods each of constant cross section.

We should reduce the problem to as few dimensions as possible. Thus a thin almost flat plate can sometimes be considered to be a section of a plane. Symmetry conditions should be used where they exist.

The simplest stress distribution compatible with the given information should be assumed, preferably a uniform or linearly varying distribution with respect to some coordinate. The same remark applies to strains, when assumptions concerning them must be made to set up the problem. We should conjecture as many stresses and strains as possible. The principle of superposition can often be employed to simplify the treatment of stresses and strains due to two or more forces and moments. Generous use should be made of Saint-Venant's principle.

Where a physical system is made up of several elastic (and perhaps rigid) bodies, we should analyze each member separately, and introduce mathematical conditions which apply at the junctions.

Since the loads are usually large compared to the weights of the parts involved, it is generally possible to neglect these weights, except where rapid dynamic phenomena occur.

Unless experience dictates otherwise, all deflections should be assumed to be small.

The density of an elastic member should be taken constant unless it is decidedly otherwise.

## REFERENCES

1. S. Timoshenko, *Theory of Elasticity*, McGraw-Hill, New York, 1934.
2. John Prescott, *Applied Elasticity*, Dover, New York, 1946.

# *Problems for Part IV*

In the problems to follow, the reader is expected to set up equations in the stresses, strains, and displacements. He is not expected to solve these equations.

1. A cable hangs under its own weight.
2. A hanging rope supports weights at two distinct points.
3. A uniform horizontal steel girder rests on a cement foundation.
4. An airplane propeller blade is subject to bending and torsion.
5. The wing of an airplane is subject to a uniform load.
6. A piano wire is struck with a hammer.
7. A horizontal continuous uniform beam rests on supports at the ends, and on two intermediate supports. The supports are equally spaced. The beam is uniformly loaded.
8. A circular plate is whirled at constant speed about an axis perpendicular to the plate through its center.
9. A rectangular plate is bent by couples into part of a right circular cylinder between two generators.
10. The axle of an automobile rests on wheels for support and is loaded at two points symmetrically placed.
11. A right circular cylindrical shaft with a flywheel at one end is subject to torsional oscillations.
12. A vertical equilateral triangle constructed from three beams hinged at the vertices forms part of a structure. The horizontal beam forming the bottom of the triangle is supported at its ends. The top vertex of the triangle is loaded.
13. Water filling a straight pipe flows through it at a constant speed. One end of the pipe is fixed. The pipe is elongated as a result of interaction between the liquid and the pipe. We are interested in this elongation.
14. The ends of a beam are subject to both bending moments and compression.
15. A railroad car with six pairs of wheels rests on a track. We are interested in the stresses and strains in the rails.
16. A horizontal uniform beam is fixed at one end, and a weight is bouncing on the other.
17. The shovel of a steam shovel is connected by a beam to the cab. At the shovel end of the beam a cable connects the cab to the beam for additional support. We are concerned with the beam.
18. A horizontal uniform beam supporting a weight at the middle is supported by two identical vertical cables at the ends. The upper ends of these cables are fixed.

19. A solid steel sphere is dropped to the bottom of the ocean.
20. A circular rod is subjected to torsion. One half of the rod is of one radius, and the other half, except for a small transition section, is of another radius.
21. A circular plate is supported at the middle, and is subjected to a uniform normal pressure.
22. The rod of problem 20 is subjected to tension.
23. The rod of problem 20 is subjected to bending moments.
24. A rod of rectangular cross section is twisted.
25. Vertical poles equally spaced carry a heavy telephone cable.
26. A vertical pendulum wire fixed at the top end has a weight on the lower end rotating about the vertical with an oscillatory motion.
27. A suspension bridge is uniformly loaded.
28. A circular rod fixed at the ends is uniformly heated.
29. The rectangular horizontal bottom of a tank is subject to the pressure of liquid above it.
30. A vertical rectangular plate with two edges horizontal forms a gate in a dam, water presses on one side of the gate, and the other is exposed to the air. The edges of the gate are in slots.
31. A helical spring is in motion under forces applied at its ends.
32. By dimensional analysis find a formula for the speed of sound along a solid rod.
33. By dimensional analysis find a formula for the stiffness (force/deflection) of a homogeneous isotropic rectangular beam.
34. A vertical right circular cylindrical bar is stretched by its own weight. By dimensional analysis find a formula for the displacement at points along the vertical center line of the bar.

# PART V

# FLUID MECHANICS

# CHAPTER

# 12

# *Fundamental fluid theory*

**12.1. Fluids.** We shall be concerned in Part V with setting up, in mathematical form, physical problems involving the flow of fluids.

By definition a body is composed of *fluid* if the body cannot support tension (tensile stress), practically speaking, and cannot support shear stresses when there is no relative motion between points in the fluid. Liquids and gases are fluids. Actually, a liquid can support some surface tension.

We consider the case where the points of the fluid are not moving relative to each other, so that like a rigid body, the fluid is not changing shape. We consider a section of the element in Figure 155, as shown in Figure 193, obtained by passing a plane through the $x$-axis making an angle $\theta$ with the $xz$-plane, where $0 < \theta < \frac{\pi}{2}$.

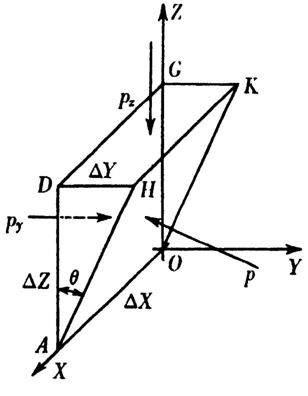


**Fig. 193. Fluid in static equilibrium.**

We let three dimensions of the section be denoted by $\Delta x$, $\Delta y$, and $\Delta z$ as shown. We designate the *average* normal stresses on the lateral faces of the prism in Figure 193 by $p_z$, $p_y$, and $p$ pointed toward the face, since these are compressive stresses. Here $p_z$, $p_y$, and $p$ are respectively vertical, horizontal, and inclined pressures as shown. We shall reverse the sign convention of Section 9.1 and take the $p$'s positive. The stresses are assumed to be continuous functions of position. We let $\Delta A$ denote the area of the face $OAHK$. The areas of the $xz$- and $xy$-faces (the latter is the face $GKHD$) are $\Delta A \cos \theta$ and $\Delta A \sin \theta$, respectively. Due to $p_y$ there is a stress component in the direction of the vector representing $p$. This component is designated by $p_y \cos \theta$, whence the corresponding component of force due to $p_y$ on the $xz$-face is given by $p_y \Delta A \cos^2 \theta$. From the face $GKHD$ we obtain the force component $p_z \Delta A \sin^2 \theta$ in the $p$ direction. From the vanishing of the resultant force in the $p$ direction

$$p_y \Delta A \cos^2 \theta + p_z \Delta A \sin^2 \theta = p \Delta A + R \Delta V, \tag{12.1}$$

where $\Delta V$ is the volume $\frac{1}{2} \Delta y \Delta A \cos \theta$ of the prism, $R$ is a function taking into account body forces and inertia, and the $p$'s are evaluated at properly chosen points. Dividing both sides of (12.1) by $\Delta A$, and taking the limit as $\Delta y$ goes to zero with $\theta$ constant, assuming that the limits exist, we obtain

$$p_y \cos^2 \theta + p_z \sin^2 \theta = p. \tag{12.2}$$

Letting $\Delta x$ go to zero simultaneously with $\Delta y$ (that is, letting the maximum of $\Delta x$, $\Delta y$, and $\Delta z$ go to zero), the relation (12.2) becomes one between the stresses at $O$ normal to the $xz$-, $xy$-, and $OAHK$-planes through $O$.

By the same mathematical limit process the components of forces perpendicular to the stress denoted by $p$ yield $p_y = p_z$. It follows that

$$p_x = p_y = p_z = p. \tag{12.3}$$

In the absence of shear stresses the components of the stress tensor at the origin are now determined. By Section 9.6 the $x$, $y$, and $z$-components of stress for any plane through $O$ are given by $lp$, $mp$, and $np$ respectively with $l$, $m$, and $n$ direction cosines of a normal to this plane. The quantities $lp$, $mp$, and $np$ are components of a normal stress denoted by $p$. Thus the normal stress is the same for all planes through $O$. This normal stress, which has the dimension $F/L^2$, is termed the *pressure* at $O$, and is often called the *static pressure*. We are considering *absolute* pressure here.

If the parts of the fluid are in relative motion in the region of space being studied, we either neglect internal forces which manifest themselves through viscosity, in which case the above theory still applies, or we *use an average p* of the normal stress in three mutually perpendicular directions as the pressure. This average can be shown to depend on the point only, and not on the directions chosen.

Due to molecular attraction tension exists at the surface of a liquid. This *surface tension* is defined in terms of forces acting on curves bounding surface elements of area just as stresses in Section 9.1 were defined in terms of forces acting on bounding surfaces of volume elements. The molecules at the surface of a liquid are not surrounded by others as in the interior, and therefore the same force balance does not exist at the surface.

**12.2. Pressure.** Pressure is often measured in pounds per square inch or square foot, and also in millimeters of mercury.

Let us suppose that a fluid is in static equilibrium under the influence of the constant force of gravity. The surface of the fluid, if there is one, may be considered a plane surface perpendicular to the direction of the force of gravity, since in the contrary case we can cut from the fluid a section as shown in Figure 194. This section is acted on only by the force $W$ of gravity and the force $F$ due to pressure. We take the plane cutting out the section to be a plane not normal to the direction of the force of gravity. The section cannot be in equilibrium because the component of the resultant force in a direction perpendicular to $W$ does not vanish. In practice, the surface of the fluid is not strictly plane, but may be considered as such provided the surface is not too large.

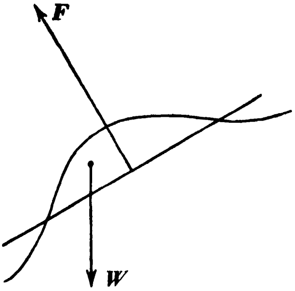


**Fig. 194. Free surface not in equilibrium.**

We consider a plane surface $S$ parallel to the surface of a fluid in static equilibrium under the influence of gravity. We shall treat the column of fluid above $S$ and shall suppose that there is no force acting on the surface of the fluid above $S$. In the absence of shear there are no vertical forces acting on the column except the weight $W$ of the column and the force due to the pressure on $S$. It follows that these are equal in magnitude.

The column of fluid need not be limited above by a surface. This may be a column of air in the atmosphere, which gradually thins out as progressively higher altitudes are reached.

A convenient reference pressure is *one normal atmosphere.* This is 14.7 pounds per square inch, or 2116 pounds per square foot, at sea level at a temperature of 15 degrees centigrade, or 59 degrees Fahrenheit. Pressure is often given in atmospheres. It is customary in work on gases to use pressure in excess of 1 atmosphere, rather than *absolute pressure;* such pressures are called *gage pressures.* Atmospheric pressure is added to gage pressure to obtain absolute pressure. Gage pressure is negative if the absolute pressure is less than 1 atmosphere, and we are said to have a *vacuum.*

We let $h$ denote the height of a mercury column in millimeters, where the pressure is zero at the top of the column. If this column is supported by a pressure $p$ applied at the base of the column, the quantity $h$ is called the *pressure head* corresponding to $p$. In general, if $\gamma$ is the weight of a liquid per cubic unit (*specific weight*), $p/\gamma$ is the height of the column of liquid that can be supported by a force due to the pressure $p$ on a unit area. The head is then $p/\gamma$ units (of length) of this liquid. Head $h$ is often given in *feet of water.* The head for one normal atmosphere is 760 millimeters of mercury.

Pressures all the way from $10^{-8}$ to $10^6$ pounds per square inch have been obtained in the laboratory, whereas liquid pressures from 1 pound per square inch to 20,000 pounds per square inch are in general use in industrial equipment, and gas pressures from 0.01 pound per square inch to 50,000 pounds per square inch.

### 12.3. Liquids and gases.

We shall distinguish between liquids and gases qualitatively. *Liquids* have extremely small compressibilities, that is, great forces are needed to change the volume of a liquid by even such amounts as 1 per cent, whereas gases are very compressible. Gases, unlike liquids, tend to expand to fill the entire available volume.

The specific *weights* (weight per unit volume) of liquids in common use vary all the way from 46 pounds per cubic foot for ether to 847 pounds per cubic foot for mercury.

If the density of air is taken as 1, the relative densities of other ordinary gases at the same physical conditions range from 0.138 for helium to 2.208 for sulfur dioxide. For mercury vapor at 100 pounds per square inch absolute, the relative density is 19.6.

Gases differ from liquids in that the molecules are widely separated compared to those in liquids, whence the densities are correspondingly smaller. Thus at 1 atmosphere and 68 degrees Fahrenheit air weighs 0.07528 pound per cubic foot, whereas under the same conditions water weighs 62.3164 pounds per cubic foot.

If pounds, feet, and seconds are used, mass $m$ is in *slugs,* as explained in Chapter 1. *Density* is then in slugs per cubic foot.

### 12.4. Ideal fluids.

A fluid that is incompressible, homogeneous (same density at each point), and nonviscous (no shear) is termed *ideal. In attacking a problem in liquids the reader should always assume that he is working with an ideal fluid unless he has reason to believe otherwise.* If there is not much relative motion between the neighboring parts of a fluid, if the conduits are large in diameter compared with the length, and if the pressures and temperatures at points in the fluid do not vary greatly, the viscosity and compressibility effects may be disregarded. The assumptions made in a particular problem depend on the nature of the problem, as examples will illustrate.

At 32 degrees Fahrenheit a unit volume of water at zero pressure goes into a volume 0.9566 under a pressure of 1,000 atmospheres, at 176 degrees Fahrenheit with zero pressure we have the volume 1.0287, and at 1,000 atmospheres the volume 0.9884.

In Section 12.18 we shall give a similar illustration for viscosity.

Gases can also often be considered ideal fluids. Thus at speeds far below the speed of sound the compressibility and viscosity of air can often be neglected in computations on flow of air past an airfoil, such as the wing or propeller of an airplane, unless considerable accuracy is required. *When gases are confined* as in pipes, *compressibility must* usually *be taken into account;* otherwise not, except for high velocities.

Let the Cartesian coordinates of a point $P$ in an incompressible, non-viscous fluid be $(x, y, z)$. Let $(a, b, c)$ be the coordinates of $P$ at some instant, say $t = 0$. Then $x$, $y$, and $z$ are functions of $a$, $b$, $c$, and $t$. This viewpoint is due to Lagrange. In terms of these coordinates we have the theorem to follow.

***Theorem* 12.1.** *At each point $P$ of an incompressible, non-viscous fluid the coordinates $x, y, z$ of $P$, the pressure $p$, and density $\rho$ are related by the differential equations*

$$\begin{aligned}\frac{\partial^2 x}{\partial t^2} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial^2 y}{\partial t^2} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y},\\ \frac{\partial^2 z}{\partial t^2} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z},\end{aligned} \tag{12.4}$$

*where $X$, $Y$, and $Z$ represent the $x$-, $y$-, and $z$-components of the body forces per unit mass at $P$, and $t$ is time.*

We can write (12.4) in the vector form

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F} - \frac{1}{\rho}\,\mathbf{grad}\ p$$

for the body force $\boldsymbol{F}$ and position vector $\boldsymbol{r}$. The equations (12.4) can be obtained from (11.1) by setting

$$\sigma_x = \sigma_y = \sigma_z = -p, \qquad \tau_{xy} = \tau_{zx} = \tau_{yz} = 0,$$

and including inertial forces in $\boldsymbol{F}$. For the benefit of the reader we shall give a direct proof of Theorem 12.1.

We consider a rectangular parallelepiped containing the point $P$, where the faces of the parallelepiped are parallel to the coordinate planes (see Figure 195). We let $\bar{p}$ be the average pressure on the back face of the element, the back face being the one where $x$ is constant and has the smallest value. The parallel face is called the *front face* in what follows. We use the term *average pressure* in the sense that the resultant force due to pressure on the back face is $\bar{p}\Delta y\Delta z$. Assuming continuity of all variables and derivatives that occur in our argument, it follows that $\bar{p}$ is the value of $p$ at some properly chosen point $P_1$ on the back face. The average pressure on the front face is $(\bar{p} + \Delta\bar{p})$, and this is the value of $p$ at some point $P_2$ on the front face. The inertial force on the element has the $x$-component denoted by $\bar{x}''\bar{\rho}\Delta x\Delta y\Delta z$, where $\bar{\rho}\bar{x}''$ is the value of $\rho x''$ at some point of the element. Here the primes designate differentiation with respect to time. By Newton's second law we have the relation

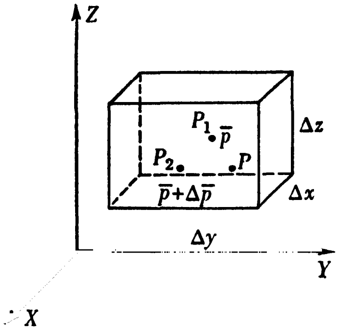


Fig. 195. Element of a fluid.

$$\bar{x}''\bar{\rho}\Delta x\Delta y\Delta z = \bar{X}\rho^*\Delta x\Delta y\Delta z - \Delta\bar{p}\Delta y\Delta z, \qquad (12.5)$$

for a properly chosen $\rho^*\bar{X}$ so that the first term on the right in (12.5) represents the $x$-component of the body force on the element.

We write

$$\Delta\bar{p} = \frac{\partial p}{\partial x}\Delta x + \frac{\partial p}{\partial y}\mu\Delta y + \frac{\partial p}{\partial z}\nu\Delta z + \epsilon \qquad (12.6)$$

for the differences $\Delta x$, $\mu\Delta y$, and $\nu\Delta z$ between the corresponding coordinates of $P_1$ and $P_2$, and a number $\epsilon$, the derivatives being evaluated at $P_1$. The number $\epsilon$ will not contribute to the final limit. If we keep $\Delta y$ and $\Delta z$ fixed and let $\Delta x$ go to zero, continuity of $p$ implies that $\mu$ and $\nu$ go to zero. We substitute the $\Delta\bar{p}$ of (12.6) into (12.5) divide by $\Delta x\Delta y\Delta z$ and take the limit as the maximum of the quantities $\Delta x$, $\Delta y$, $\Delta z$ goes to zero, while the element shrinks to $P$. We suppose that the limits of $\mu/\Delta x$ and $\nu/\Delta x$ exist as $\Delta x \to 0$. Mathematical continuity considerations yield (12.4).

Theorem 12.1 applies to ideal fluids. It is to be observed that an incompressible fluid need not be homogeneous. The density $\rho$ at a point $P$ of the fluid is in general a function of $a$, $b$, $c$, and $t$. For incompressible fluids $\rho$ is independent of $t$, but may vary with $a$, $b$, and $c$.

### 12.5. Continuity equation for incompressible fluids.

The equations (12.4) form a system of three equations in the unknown functions $x$, $y$, $z$, and $p$ of the time $t$. It is evident that we need another equation to determine these functions. This equation follows from the indestructibility of matter, which implies that the rate at which fluid enters a region $R$ of space is equal to the rate at which fluid leaves $R$. We are supposing that we have no sources or sinks, meaning points at which fluid suddenly appears or disappears.

For practical purposes it is often convenient to introduce sources and sinks, as when we treat a reservoir with an outlet through a hole in the bottom. The hole, if not too large compared with the dimensions of the reservoir, could be taken to be a point at which the fluid is destroyed, that is, a sink. It is a simple matter to alter the following equations to take care of sources and sinks.

We are thus using the principle that follows.

***Principle* 12.1.** *Mass can neither be created nor destroyed.*

This principle is called the *continuity principle*. It applies even in nuclear physics, as for example in the case of the atom bomb, if energy equivalent to mass is taken into account.

In general no new assumptions are needed to treat the theory of fluids, Newton's second law of motion and the laws of conservation of energy and mass being sufficient to cover the problems that arise in engineering. Following Euler, we let $x$, $y$, $z$, and $t$ be the independent variables in the theorem below.

***Theorem* 12.2.** *If $u$, $v$, and $w$ represent the $x$-, $y$-, and $z$-components of the velocity $V$ of a point $P(x, y, z)$ in a homogeneous, incompressible fluid, these components satisfy*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{12.7}$$

Equation (12.7) is equivalent to

$$\operatorname{div} V = 0.$$

Theorem 12.2 is the *divergence theorem* applied to fluids. The rate at which fluid crosses the back face of the parallelepiped used in the proof of Theorem 12.1 is $\rho\bar{u}\Delta y\Delta z$ for the $x$-component of velocity $\bar{u}$ evaluated at a properly chosen point $P_1$ on this face, and density $\rho$, it being assumed that the velocity varies continuously over the element.

The $v$ and $w$ components contribute nothing to the flow across the back and front faces. The rate at which fluid is flowing across the front face is $\rho(\bar{u} + \Delta\bar{u})\Delta y\Delta z$ evaluated at some point $P_2$ on this face.

The net flow rate across the back and front faces is $\rho\Delta\bar{u}\Delta y\Delta z$. Consideration of all the faces yields the net rate of flow, which vanishes, whence

$$\Delta\bar{u}\Delta y\Delta z + \Delta\bar{v}\Delta x\Delta z + \Delta\bar{w}\Delta x\Delta y = 0, \tag{12.8}$$

where $\Delta\bar{v}$, and $\Delta\bar{w}$ come from properly chosen points. We can write $\Delta u$, $\Delta v$, and $\Delta w$ in expanded form as for $\Delta\bar{p}$ in (12.6), and substitute the results in (12.8). Dividing by $\Delta x\Delta y\Delta z$, and taking the limit as the maximum of the quantities $\Delta x$, $\Delta y$, $\Delta z$ goes to zero, the relation (12.8) yields (12.7), assuming that the limits exist.

*Theorems 12.1 and 12.2 with boundary conditions completely describe the motion of an ideal fluid.* For compressible and viscous fluids these equations can be modified in simple ways where necessary.

As was true for fields in electrical engineering, it is not generally convenient to work with the partial differential equations (12.4) and (12.7). As we shall see, simplifying assumptions are usually made to give practical results quickly.

**12.6. Flow of water through a wide pipe.** In many cases the equations (12.4) and (12.7) reduce to very simple form. As an example we shall consider water flowing through a horizontal right circular cylindrical pipe 10 feet long, 1 foot in diameter. At the left end of this pipe a pressure of 100 pounds per square inch is maintained, and at the right end the water empties into a standard atmosphere. It is assumed that the water has been flowing long enough so that a steady state prevails.

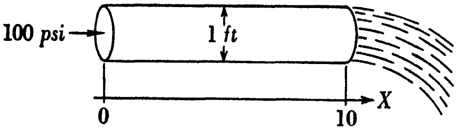


**Fig. 196. Flow through a horizontal pipe.**

We set up an $x$-coordinate axis as shown in Figure 196. Our units will be pounds, feet, and seconds. The equations (12.4) and (12.7) now become

$$\frac{dp}{dx} = 0, \qquad \frac{du}{dx} = 0. \tag{12.9}$$

We have the *boundary* conditions

$$x = 0, \quad p = 14{,}400; \qquad x = 10, \ p = 2{,}116. \tag{12.10}$$

The equations (12.9) give

$$p = C_1, \qquad u = C_2 \tag{12.11}$$

for constants $C_1$ and $C_2$. The first relation of (12.11) yields a *contradiction* with the boundary conditions for $p$ of (12.10). It follows that our analysis has been much too simplified; in fact, there can be no drop in pressure across the pipe when we treat the water as an ideal fluid.

We shall, therefore, consider a different problem and leave the solution of the above example until later when we take into account viscosity, which clearly cannot be neglected in the above problem.

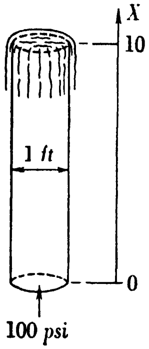


**Fig 197. Flow out of a vertical pipe.**

We shall instead take the pipe of Figure 196 vertical as shown in Figure 197 so that the body force of gravity enters into the picture. The equations (12.4) and (12.7) now become

$$\frac{dp}{dx} = -62.3, \qquad \frac{du}{dx} = 0. \tag{12.12}$$

The first term on the right of (12.12) is obtained from the fact that $\rho X$ is the body force (weight) per unit volume. We still have the boundary conditions of (12.10). By (12.12)

$$p = -62.3x + C_1, \qquad u = C_2 \tag{12.13}$$

for constants $C_1$ and $C_2$. We can determine $C_1$ so that the first condition of (12.10) is satisfied, but the second is not then satisfied. Again our mathematical model is *inconsistent*, and viscosity cannot be neglected.

We shall, therefore, *modify the problem once again* and suppose that the second condition of (12.10) is given, that is, the pipe empties into the atmosphere, and that we have

$$x = 10, \qquad u = 10,$$

that is, the velocity of the water on leaving the pipe is 10 feet per second. The initial conditions, and (12.13), now yield the solutions

$$p = 2{,}739 - 62.3x, \qquad u = 10. \tag{12.14}$$

Thus velocity and pressure are everywhere determined in the conduit, the velocity being in the $x$-direction only and constant at 10 feet per second, and the pressure varying linearly with $x$. In particular the pressure at the bottom of the pipe is 2,739 pounds per square foot. The problem is thus consistent.

**12.7. Transient flow through a vertical pipe.** In the problem of this section there is a lid on the top of the pipe of Figure 197 at the start $t = 0$. At $t = 0$ the fluid is at rest. The pressure at $x = 0$ is kept at 100 pounds per square inch. The lid is instantly removed.

Equations (12.4) and (12.7) now give

$$\frac{\partial^2 x}{\partial t^2} = -32.2 - \frac{32.2}{62.3}\frac{\partial p}{\partial x}, \qquad \frac{\partial u}{\partial x} = 0, \tag{12.15}$$

and the initial and boundary conditions are

$$x = 0, \quad p \equiv 14{,}400; \qquad t = 0, \quad u \equiv 0; \qquad x = 10, \quad p \equiv 2{,}116. \tag{12.16}$$

Solution of the second equation in (12.15) yields

$$u = f(t) \tag{12.17}$$

for a function $f(t)$ of $t$. Since the motion is one-dimensional, the derivative $\partial^2 x/\partial t^2$ in (12.15) is simply $du/dt$. Thus the first equation in (12.15) becomes

$$\frac{\partial p}{\partial x} = -62.3\left[1 + \frac{f'(t)}{32.2}\right]. \tag{12.18}$$

The function on the right of (12.18) is a function of $t$ only, whence

$$p = h(t)x + k(t) \tag{12.19}$$

for functions $h(t)$ and $k(t)$ of $t$. From the first and last conditions of (12.16) we have

$$p = -1{,}228.4x + 14{,}400. \tag{12.20}$$

By (12.20) and (12.18)

$$f'(t) = 603. \tag{12.21}$$

By (12.17) and (12.21)

$$u = 603t + C \tag{12.22}$$

for a constant $C$. By (12.22) and the second condition of (12.16) we have $C = 0$. Integration of (12.22) gives

$$x = 302t^2 + C^*, \tag{12.23}$$

where we have rounded off $\frac{603}{2}$. The value of $C^*$ is determined by the initial position of the point in the fluid being studied. Thus $C^*$ is zero for all points at $x = 0$ when $t = 0$. The formula (12.23) then gives the position of such points at the time $t$.

The complete solution of our problem is thus given by (12.20) and (12.23). The fact that we arrived at a solution indicates that our assumption that we were working with an ideal fluid was justified.

It is to be observed that the relations of (12.16) are conditions on functions, giving the values of these functions for all values of one variable $x$ or $t$, the other variable being constant. This is quite different from what happens for ordinary differential equations. The reader should familiarize himself with the methods of solution of partial differential equations so that he will know what boundary conditions are needed to determine his problem.

**12.8. Pressure energy.** Because it is generally too difficult to work with the differential equations (12.4) and (12.7), or their analogues for the compressible and viscous cases, it is more convenient to use energy relations as in Section 2.18.

We shall employ the concept of *energy density*, that is, we shall introduce an energy point function which by integration yields the energy in a region of the fluid, just as the mass can be secured by integrating the mass density. In engineering, energy density is termed *head*. In particular, there is a pressure head dependent on pressure only. It is logical to take this as zero when pressure is zero, which is done.

We shall derive the pressure head at a point $P$ of a fluid at which the pressure is $p$. It is no restriction on the generality of the method to imagine that the fluid is contained in a body of fluid unlimited in horizontal extent and under the influence of a constant force of gravity. The body is assumed to have a surface at $x = 0$, the $x$-axis being vertical and pointed downward. In our argument the fluid may be viscous, since the fluid moves as a unit and viscosity effects do not occur. The fluid is taken incompressible. It is a simple matter to extend the argument to compressible fluids.

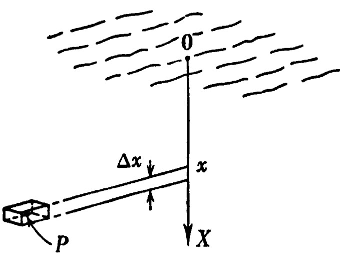


Fig. 198. Flow with a surface at $x = 0$.

We enclose $P$ with a rectangular parallelepiped of vertical sides as shown in Figure 198. The height of the element is taken to be $\Delta x$, and the area of a horizontal face is taken to be unity.

The pressure on the top face of the element is equal to the weight of the liquid above it, which is $\rho gx$ for the density $\rho$ of the liquid, the magnitude $g$ of the acceleration of gravity, and coordinate $x$ of the top face. Because of pressure on the top face, it takes a certain amount of work $W_t$ to bring the element to the surface of the liquid where $x = 0$, the work $W_t$ being

$$W_t = \rho g \int_0^x x dx, \tag{12.24}$$

namely

$$W_t = \tfrac{1}{2}\rho g x^2. \tag{12.25}$$

The pressure on the bottom face of the element is $\rho g(x + \Delta x)$, so that the work $W_b$ done in bringing the element to $x = 0$, due to the pressure on the bottom face only, is

$$W_b = \tfrac{1}{2}\rho g(x^2 + 2x\Delta x).$$

We let $W = W_b - W_t$. The difference $W$ is then $\rho gx\Delta x$. Dividing $W$ by the volume of the element we have $\rho gx$ as the net work $W_u$ per unit volume, due to pressure, necessary to bring the element to the top. Clearly, $W_u$ is the pressure on the top face of the element in the initial position.

Taking the limit as $\Delta x$ goes to zero we obtain $p$ as the pressure energy per unit volume at the point $P$. Let $\gamma$ be the specific weight of the fluid. It is customary to use the density $p/\gamma$, instead of $p$, this being the pressure energy per unit weight. The density $p/\gamma$ is called the *pressure head.*

For compressible fluids the above proof applies except that the element will change geometrically in being brought to the top.

If the fluid has no surface, as for a free gas in space, the above proof applies with the modification that we must work with limits of integrals. We must treat the fluid as compressible in this case, for otherwise the pressure would be infinite at each point.

The element in the discussion above might be at the bottom of a vertical cylindrical tube with the top exposed to the air. By scooping off the liquid at the top of the tube, doing little or no work in the process, we can make the element rise to the surface. This element rises to the surface under the influence of the neighboring fluid, and the pressure energy originally associated with the element is converted into work.

In Part III we defined *flow work.* This is a difference between values of the pressure energy. Here we have arrived at this difference from a slightly different point of view.

**12.9. Velocity and potential heads.** At each point of a fluid there is an energy density due to the velocity of the fluid.

In the case of any fluid the mass of an element of volume $\Delta V$ is $\rho\Delta V$, where $\rho$ is the value of the density at some point in the element, and is continuous over the element. Assuming that the velocity is also continuous, the kinetic energy possessed by the element is $\frac{1}{2}\rho v^2 \Delta V$, where $v$ is the speed at some point in the element. The average kinetic energy density per unit weight ($\gamma = \rho g$) is then, by definition, $v^2/2g$ for the element. Taking the limit as the maximum linear dimension of the element goes to zero and the element shrinks to the point $P$, we obtain the *kinetic energy density* $v^2/2g$ per unit weight at the point $P$, the speed of the fluid at $P$ being $v$. This density is also termed the *velocity head*, and is usually given in feet of water or millimeters of mercury. See Section 6.8.

Let a point $P$ in a fluid be a height $z$ above a reference plane. From the technique of taking an element of volume containing $P$ as above, and going to the limit as the element shrinks to $P$, it is seen at once that there is an energy density (gravitational head) due to the fact that in moving the element vertically a distance $z$ under a constant force of gravity, work is done equal to the weight of the element times the distance a point in the element is moved. The gravitational head, called *potential head*, is thus $z$ itself.

If we are treating a problem where only forces due to pressure, gravitation, and inertia are considered, the energy densities involved are those above. If, however, chemical transformations or certain other phenomena occur, such as temperature changes, other forms of energy density must normally be taken into account to satisfy the law of the conservation of energy.

For energy density (as energy per unit weight) the law of conservation of energy takes the form of Bernoulli's theorem (below) when the motion is steady. The fluid motion is said to be *steady* when

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0$$

for the velocity components represented by $u$, $v$, and $w$. Here $u$, $v$, and $w$ are functions of $x$, $y$, $z$ and $t$. In the case of steady motion we have a *velocity field*, given by a region of space and a velocity at each point of the region. A *line of motion* is by definition a curve for which

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

The velocity is in the direction of a tangent ray to the line of motion. By mathematical continuity assumptions involving $u$, $v$, and $w$, and differential equations theory, we can show that the lines of motion are determined by the velocity field, and that there is only one line of motion through each point. In this event each *particle* of fluid describes a *streamline* with the property that this is identical with a line of motion. In the theorem to follow we refer to such motion. A streamline is by definition a curve such that at any instant the velocity of each *particle* of fluid on the stream line is along the tangent to this line. When considering only pressure, velocity, and potential heads *Bernoulli's theorem* states that *for steady motion of an ideal fluid the total energy density, that is, head, is a constant for the points of a streamline.*

A proof of Bernoulli's theorem will be given below. When treating energy in general this theorem should be assumed to be valid, including terms to take care of heat, work, friction, chemical, and other forms of energy that may be involved. Thus in its general form Bernoulli's theorem is not a theorem, but the principle of the conservation of energy in a certain form.

We consider a plane region $R$ of area $A$ perpendicular to a streamline $L$, where $L$ passes through $R$. We suppose that $L$ has an arc length coordinate $s$, where $s$ increases in the direction of flow. In what follows $t$ is the independent variable. The particles at $R$ at the instant $t$ travel some *average* distance $\Delta s$ during the time $\Delta t$ after the instant $t$. At the instant $(t + \Delta t)$ the region $R$ has gone into a region $R^*$ of area $A^*$, where $A$ is a good approximation to $A^*$ when $\Delta t$ is small. The force due to pressure on the region $R$ is given by $pA$ for some properly chosen pressure $p$. There is a corresponding component of the pressure force on the region $R^*$, where this component is given by $(p + \Delta p)A^*$ for some properly chosen increment $\Delta p$. During the interval $\Delta t$ the particles at $R$ at the instant $t$ sweep over a region $M$ of space between the surface regions $R$ and $R^*$. The net pressure force on $M$ perpendicular to $R$ is then approximately $-A\Delta p$ at least when $\Delta t$ is small enough.

Let $\rho$ be the density of the fluid. We shall treat the case where $\rho$ is constant for practical purposes. We let $z$ be the vertical coordinate of a point on $L$ with the positive direction upward. The force of gravity acting on the fluid $M$ is given approximately by $Ag\rho\Delta s$, and its component in the direction of the streamline $L$ at $R$ is this force multiplied by $-dz/ds$.

Let $V$ represent the velocity of the fluid at a point of $L$ so that V is $ds/dt$. The tangential component of acceleration is given by $dV/dt$, which is the product of $dV/ds$ by $ds/dt$. Since $V$ is $ds/dt$, the tangential component of acceleration is represented by $VdV/ds$. The mass of the fluid in the region $M$ times this acceleration is then approximately $(A\rho\Delta s)VdV/ds$. From Newton's second law of motion we have approximately

$$(A\rho\Delta s)V\frac{dV}{ds} = -A\Delta p - \frac{dz}{ds}(Ag\rho\Delta s).$$

Dividing by $A\Delta s$, and taking the limit as the maximum length of the chords joining points of the region $M$ goes to zero, we have

$$\rho V\frac{dV}{ds} = -\left(\frac{dp}{ds} + g\rho\frac{dz}{ds}\right).$$

The approximations then disappear. It is assumed of course, that the limits exist. The last relation yields

$$\frac{V^2}{2g} + z + \frac{p}{\gamma} = C$$

for the specific weight $\gamma = \rho g$ and a constant $C$. The constant $C$ depends on the streamline. If the subscript 1 denotes the values of the variables at one point on a streamline, and the subscript 2 denotes these values at another point,

$$\frac{V_1^2}{2g} + z_1 + \frac{p_1}{\gamma} = \frac{V_2^2}{2g} + z_2 + \frac{p_2}{\gamma}.$$

**12.10. Flow through an orifice.** By definition an orifice is an opening in a surface bounding a fluid. In practical examples an orifice is a *hole* short in length compared to the maximum diameter taken normal to the *average* direction of flow. In many cases the velocity of discharge is so great compared to the velocity of the fluid entering the orifice, that the latter velocity is neglected. Since an orifice is short in length, the potential head is disregarded. Otherwise the orifice might be treated as a pipe.

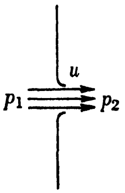


**Fig. 199. Flow through an orifice.**

In Figure 199 we have pictured an orifice with pressure $p_1$ on one side and $p_2$ on the other. The difference $h$ equal to $(p_1 - p_2)/\gamma$, for the specific weight $\gamma$, is termed the *pressure head* for the orifice, this being actually a difference between heads. For the standard orifice case we have the theorem that follows.

***Theorem* 12.3.** *If $h$ is the pressure head across an orifice, $g$ the magnitude of the acceleration of gravity, and zero the speed of the fluid entering the orifice, the speed $u$ of efflux is given by*

$$u = \sqrt{2gh} \tag{12.26}$$

*for an ideal fluid.*

By Bernoulli's theorem

$$\frac{p_1}{\gamma} = \frac{p_2}{\gamma} + \frac{u^2}{2g} \tag{12.27}$$

for the specific weight $\gamma$. Relation (12.26) follows at once. Theorem 12.3 is due to *Torricelli* who stated that the speed $u$ of efflux is equal to the speed of a body falling in a vacuum a distance of $h$ feet from rest.

If the speed $u_1$ of the fluid entering the orifice is not small compared to $u$ we have the term $u_1^2/2g$ added to the left member of (12.27), whence (12.26) becomes

$$u = \sqrt{2gh + u_1^2}. \tag{12.28}$$

In a practical example the velocity across the orifice will not be uniform, whence (12.26) is replaced by

$$u = C_v \sqrt{2gh} \tag{12.29}$$

for the *coefficient of velocity* $C_v$. For orifices about an inch or more in diameter the coefficient $C_v$ for water is very nearly unity. The constant $C_v$ will vary somewhat with the head.

With orifice area $A_0$ the volumetric discharge velocity $Q$ is $A_0u$ in the ideal case for $u$ of (12.26), but in practice we have

$$Q = CA_0 \sqrt{2gh} \tag{12.30}$$

for the *coefficient of discharge* $C$. A typical value of $C$ for water is 0.6. It is customary to use the *effective area* $A$ in place of the actual area so that the formula

$$Q = A \sqrt{2gh} \tag{12.31}$$

is employed in place of (12.30).

**12.11. Emptying a reservoir.** We consider the reservoir of Figure 200 with constant cross section of area $10^3$ square feet, and with the water level 30 feet at the start. A valve with effective area 1 square foot is suddenly opened so that the water discharges into the atmosphere.

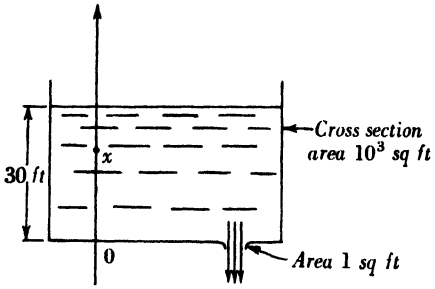


Fig. 200. Emptying reservoir.

We set up an $x$-axis as shown, $x$ being the instantaneous height of the water. The pressure on the bottom of the reservoir at the hole is thus $62.3x$ pounds per square foot. We are neglecting the pressure of the air which is practically the same on the bottom of the hole as on the surface of the water above the hole, so that the effect of this pressure cancels out. We also neglect the initial transient. By (12.31) with $h = x$ we have the equation

$$Q = 8\sqrt{x}. \tag{12.32}$$

Here we have taken $g = 32.2$ ft per sec$^2$ as usual. However,

$$Q = -10^3 \frac{dx}{dt}, \tag{12.33}$$

whence

$$\frac{dx}{dt} = -0.008\sqrt{x}. \tag{12.34}$$

Equation (12.34) is a nonlinear ordinary differential equation of the first order. To solve it we need the initial condition

$$t = 0,\ x = 30. \tag{12.35}$$

**12.12. Solenoid valve problem.** We shall treat a solenoid valve as shown in Figure 201. In this figure $R$ and $L$ are the resistance and inductance of the coil, $i$ is the current through the coil and $e$ the corresponding voltage, $p_0$ is the oil supply pressure, $k_1$ is the scale of the spring loading the plunger, and $x$ is the coordinate of position of the plunger measured from the just closed position. The oil flows from the supply through the valve to a cylinder with piston and connecting parts of mass $m$, and position coordinate $z$. The piston is assumed to be loaded by a spring with scale $k_2$. The pressure on the right side of the piston is atmospheric, and on the left side the pressure is a variable $p$.

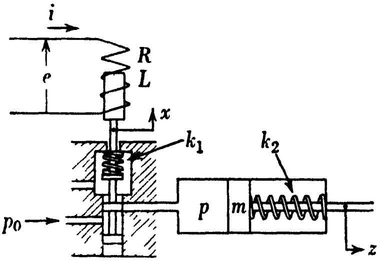


**Fig. 201. Solenoid valve controlling oil flow to a cylinder.**

See Figure 202 for a cross section of an actual solenoid valve.

We shall treat the problem where the valve is in the just closed position, and the whole system is in static equilibrium. The value of $e$ is then suddenly changed by an amount $E$ to open the valve. Except for $p$, which is absolute pressure, we can measure our variables and forces as deviations from the static equilibrium where the pressure drop across the piston is zero.

For a normal valve the volumetric rate of flow of the oil through the valve increases with the valve opening. It is therefore reasonable, as a first try, to assume that

$$\frac{dz}{dt} = Kx\sqrt{p_0 - p} \tag{12.36}$$

for a positive constant $K$ and time $t$. Equation (12.36) follows immediately from (12.31) and the fact that the volumetric rate of flow of oil into the cylinder is the cross-section area of the piston times the speed of the piston. We are neglecting leakage and compressibility effects.

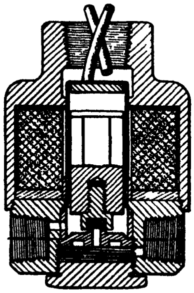


**Fig. 202. Cross section of a solenoid valve. Courtesy of the Barber-Colman Company.**

The quantities $e$ and $i$ are related by the equation:

$$Ri + L\frac{di}{dt} = e. \tag{12.37}$$

We shall suppose that the plunger is below the center of the solenoid when $x = 0$, and that during disturbances it never goes beyond the center. For a given $i$ we would expect the permeability of the core of the coil to increase as the plunger approaches the center position, and to decrease otherwise. By the flux treatment of Chapter 3 the force on the plunger will increase with $x$. The force will be represented by $F$. The force $F$ will also increase with the current in the coil. Since we can use differentials in place of deviations, if the deviations are not too large, we can approximate the deviation $F_p$ of $F$ from equilibrium by

$$F_p = c_1 i + c_2 x \tag{12.38}$$

for positive constants $c_1$ and $c_2$.

We shall write the mass of the plunger as $m_p$. From Newton's second law of motion

$$m_p x'' = c_1 i + c_2 x - k_1 x \tag{12.39}$$

for the acceleration $x''$. In actual practice we must also take account of the hydraulic reaction at the valve. We shall assume that this reaction is included in the spring scale. In any event (12.39) can be written for a constant $k$ as

$$m_p x'' + kx = c_1 i. \tag{12.40}$$

We let $A$ be the cross-section area of the piston, and $p_a$ the pressure of the atmosphere. By Newton's second law

$$mz'' = (p - p_a)A - k_2 z. \tag{12.41}$$

In deriving (12.41) we have neglected friction, which is ordinarily justified.

We have a system of four differential equations (12.36), (12.37), (12.40), and (12.41) in the four variables $x$, $z$, $i$, and $p$ as functions of time $t$.

It remains to obtain the initial conditions. We shall consider (12.37) first. When $t < 0$, $e = i = i' = 0$. Since $i$ must change continuously, we have the initial condition

$$t = 0,\ i = 0. \tag{12.42}$$

Equation (12.37) and the condition (12.42) completely describe the solution for the current $i$. With this $i$ we can solve (12.40) for $x$ provided we know the values of $x$ and $x'$ when $t = 0$. These are

$$t = 0,\ x = 0,\ x' = 0. \tag{12.43}$$

With $x$ known in terms of $t$ we consider the solution of (12.36) and (12.41) for $p$ and $z$. We can solve (12.41) for $p$ in terms of $z$ and $z''$, and substitute in (12.36) obtaining a nonlinear differential equation, which can be written as a linear polynomial in $z''$, $(z')^2$, and $z$ equal to zero, the coefficients being functions of $t$.

By (12.36), (12.43), and $z = 0$ at the start

$$t = 0,\ z = 0,\ z' = 0, \tag{12.44}$$

whence the solution for $z$ is determined. In view of (12.41) the solution for $p$ is also determined, whence our problem is solved.

If friction and leakage are to be taken into account the equations (12.36), (12.39), and (12.41) must be modified by corresponding additional terms.

## 12.13. Jet of water striking a spring-loaded plate.

We shall treat the case of a jet of water with varying speed $v$ striking against a plate as in Figure 203. The plate is supported by a light steel spring with scale $k$. We introduce a coordinate $x$ for the plate, where the plate is in the position $x = 0$ when the water is not flowing and the state is stationary. We take the positive sense on the $x$-axis in the direction of contracting spring.

We shall take the effective area $A$ of the nozzle to be 2 square inches, and the scale $k$ at 100 pounds per inch. Let us suppose also that $v$ is in a reasonable range, say from zero to 1,000 inches per second, and that the given area of the plate is about 10 square inches, while the plate is very thin but strong enough so as to be a rigid body for practical purposes. We assume that the plate dimensions are large enough so that the entire jet strikes the plate.

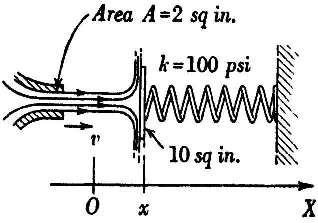


**Fig. 203. Jet of water striking a spring-loaded plate.**

It is reasonable to neglect the mass of the plate and the spring as a first approximation, since the applied forces are normally large by comparison with the inertia of these parts.

It is also reasonable to suppose that on striking the plate the jet is deflected parallel to the plate surface away from the region of impact as shown.

The $x$-velocity of the water is changed to that represented by $x'$ on impact with the plate, whence the decrease in velocity of the fluid at impact is given by $(v - x')$.

The mass rate $\rho Q$ of fluid striking the plate is $\rho A(v - x')$ for the density $\rho$ of the water. With pounds, inches, and seconds the density $\rho$ is $935 \cdot 10^{-7}$. The rate of change of momentum of the water at the plate is the product of the change in the velocity of the fluid on impact by the mass rate of flow, and is thus represented by $0.000187(v - x')^2$ if $v > x'$. Newton's second law of motion, stated in the more general form for momentum, gives

$$0.000187(v - x')^2 = 100x. \tag{12.45}$$

If we have an oscillatory case, where for example $v = 1 + \sin t$, the solution for $x$ is determined by (12.45) and the initial value of $x$. If we are interested only in the steady-state solution we do not need this initial value of $x$, on the assumption that the initial transient dies out rapidly so that it can be neglected.

If $v < x'$ the jet does not strike the plate, which case could not arise in our present treatment because then we would have the spring force unbalanced.

As we have taken our problem, the coordinate $x$ is always positive. The velocity $v$ is also positive. We did not need to use the area of the plate explicitly.

### 12.14. Checking a solution by weakening assumptions.

The remarks of this section apply to all fields of analysis treated in this book, but will be illustrated in the next two sections by a hydraulic example. In cases where experience or experiments are not available, it is advisable to take neglected factors into account. It would normally be expected that the *larger the neglected factors, the larger are the deviations from the results based on simplifying assumptions.* It is therefore advisable, if the neglected factors are not exactly known, to exaggerate these factors one by one and study the effect on the solution. In calculus, through the theory of total differentials, errors in a function $f$ of two or more variables are treated by computing the approximate errors $\{df\}$ in $f$ due to the errors in each of these variables individually, and obtaining the total error in $f$, approximately, by adding the errors $\{df\}$. Similarly, mathematical considerations show that it is reasonable in the standard situation to suppose that the following principle holds.

***Principle* 12.2.** *If the deviation of the solution caused by taking into account neglected factors individually is negligible, the deviation of the solution for these factors taken into account simultaneously is negligible.*

### 12.15. Example of weakening assumptions in an oscillatory case.

We shall illustrate Principle 12.2 with the example of Section 12.13. We shall consider the solution for the oscillatory case where $v = 1 + \sin t$.

Suppose that the plate and spring together weigh 1 pound, whence the mass $m$ of these parts is $\frac{1}{386}$, that is 0.00259. Assuming that the entire mass is concentrated at the target, the differential equation (12.45) becomes

$$0.000187(v - x')^2 = 0.00259x'' + 100x \tag{12.46}$$

when this mass is taken into account. We note that if $x = \sin t$ the first term on the right in (12.46) is negligible compared to the second term. This suggests that with $v$ given as a periodic function of $t$ with the same frequency as $\sin t$ the first term is still negligible compared to the second. Actual solution, or a study of the properties of Fourier series approximations to the solution, shows that this is correct.

In Section 12.13 we have also neglected friction in the spring, and air resistance to movement of the plate. It is a simple matter to compute the air resistance to the plate, just as we computed the hydraulic force on the plate due to the change of momentum of the fluid in Section 12.13. The friction in the spring is not such an easy matter, and here we should fall back on experience or measurement. Such studies will show that the damping-friction force can be approximated by $-cx'$ with $c$ certainly much less than 0.01, so that (12.45) becomes *at worst*

$$0.000187(v - x')^2 = 0.01x' + 100x \tag{12.47}$$

when damping and friction are taken into account.

The same argument as used for $m$ above shows that the first term in (12.47) is negligible compared to the second for the oscillatory case where $v = 1 + \sin t$. It follows that both mass and damping are negligible, as assumed in Section 12.13.

Consider now the case of Section 12.13 where

$$v = 1 + \sin 300t. \tag{12.48}$$

The first term on the right of (12.46) is now no longer negligible. The first term on the right in (12.47) is still negligible.

**12.16. Weakening assumptions in a transient case.** Let us consider the problem of Section 12.13 where from a static equilibrium with $v = 0$ the speed of the water is instantly (for practical purposes) increased to $v = 10$ inches per second as by opening a valve.

If we use (12.45) for the plate-spring system, neglecting damping and inertia, it follows that $x'$ changes instantly from zero to 10 inches per second. This implies infinite acceleration $x''$ of the plate at the start, which is impossible. It is clear that the mass $m$ should probably not be neglected. A practical solution can be obtained from (12.46) with $v = 10$, and the initial conditions

$$t = 0,\ x = 0,\ x' = 0. \tag{12.49}$$

In (12.46) we should really use one-half the weight of the spring in place of the entire weight, for greater accuracy.

**12.17. Impulse wheel.** We shall derive the differential equation of an impulse waterwheel (Figure 204). For such a wheel, jets of water are directed at buckets on the wheel (see Figure 205), and as in Section 12.13 the rate of change of momentum of the water creates a force on the buckets and hence a torque $T$. We shall treat the case where the volumetric rate of flow $Q$ of the water striking the buckets is increased suddenly from zero for a static state to a fixed value $Q_0$. We shall suppose that the velocity of the water is constant. The power absorbed by the wheel is proportional to $Q$.

Let $N$ be the speed of the wheel. The power consumed by the wheel is the product $TN$, whence the torque $T$ is proportional to $Q/N$. If $T$, $Q$, and $N$ do not vary over too wide a range, it is reasonable to use the differentials of these variables instead of the variables. Now letting $T$, $Q$, and $N$

denote the differentials of torque, rate of flow, and speed, respectively, we have

$$T = aQ - bN \tag{12.50}$$

for positive constants $a$ and $b$. Equating $T$ to the inertial torque $IN'$ we obtain the differential equation of motion of the wheel. The initial condition is

$$t = 0, N = 0.$$

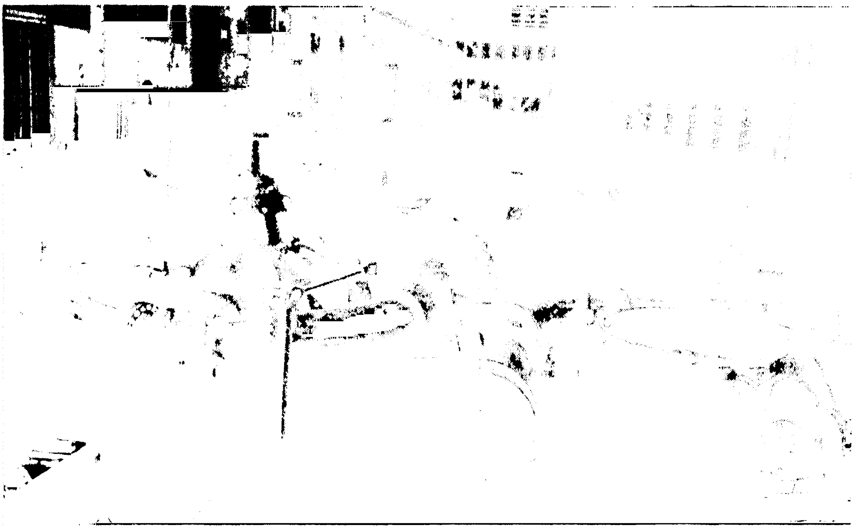


Fig. 204. Two 4-nozzle vertical impulse turbines on an erecting floor. 13,300 horsepower. 600 rpm. 1301 ft head. Courtesy of the Pelton Waterwheel Company.

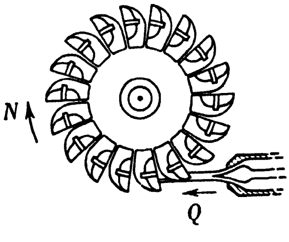


Fig. 205. Impulse wheel.

**12.18. Viscosity.** To treat viscosity of fluids the theory of ideal fluids is modified to satisfy the two laws which follow.

***Law* 12.1.** *Whenever a fluid is in contact with a solid wall, and the fluid wets the wall, there is no motion of the fluid relative to the wall at the points of contact.*

Thus when the fluid wets the wall it does not slip past the wall.

In the following law $x$, $y$, and $z$ refer to a Cartesian coordinate system.

**Law 12.2.** (**Newton**). *If $u$ and $v$ represent the $x$- and $y$-components of the velocity of a point $P$ in a fluid, the shear stress $\tau_{yx}$ at $P$ is such that*

$$\tau_{yx} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{12.51}$$

*for the coefficient of viscosity $\mu$.*

The quantity in the parentheses in (12.51) is the rate of change of the shear $\gamma_{xy}$, so that (12.51) can be written as

$$\tau_{yx} = \mu \frac{\partial \gamma_{xy}}{\partial t},$$

which replaces the stress-strain relation

$$\tau_{yx} = G\gamma_{xy}$$

for elastic bodies. See Sections 9.1 and 9.3 for the definitions of $\tau_{yx}$ and $\gamma_{xy}$.

The coefficient of viscosity $\mu$ depends on the temperature of the fluid as well as other factors which can normally be neglected.

The coefficient of viscosity $\mu$ is often given in centipoises. A *centipoise* is 0.01 poise, and is the viscosity of water at 68.4 degrees Fahrenheit and one atmosphere in the metric system of grams, centimeters, and seconds. The poise is the resistance, in dynes per square centimeter of its surface, offered by a layer of the fluid to the steady motion parallel to that layer, of another layer of the fluid at a distance of 1 centimeter from it, with a relative speed of 1 centimeter per second. At 32 degrees Fahrenheit the quantity $\mu$ in centipoises is 1.792 for water, and at 212 degrees Fahrenheit it is 0.284. For each temperature the viscosity $\mu$ of water is practically constant for the pressure ranges that arise in engineering.

To change from the metric system to the pounds, feet, seconds system multiply centipoises by $2.08 \cdot 10^{-5}$, giving slugs per foot per second. The dimensions of $\mu$ are $M/LT$, or $FT/L^2$ for $F$ denoting force.

For air at 32 degrees Fahrenheit and a pressure of one atmosphere the viscosity $\mu$ is 0.0171. For a heavy lubricating oil $\mu$ may be as high as 4,000 centipoises. Also, $\mu$ for gases is relatively independent of pressure in usual engineering pressure ranges (up to 100 atmospheres for many

Viscosity is due to internal fluid forces.

In treating flow problems it is convenient to distinguish between two types, *laminar* and *turbulent*. In the case of laminar flow the motion is steady, or practically so, and the fluid travels in layers or *laminae*. Even though we may have turbulent flow, as between the blades of a gas turbine compressor, we can still obtain results accurate enough for many purposes by supposing that the flow is laminar. Reynolds introduced the number $R$ given by $\rho Vl/\mu$ for the density $\rho$, speed $V$, viscosity $\mu$, and some characteristic dimension $l$ of the bounding surfaces (see Section 8.16). If $R$ is small the flow is laminar, and if large, it is turbulent as verified, for example by flow through pipes. Whether or not $R$ is small or large depends on the problem. However, in general, for a circular pipe with $l$ equal to pipe diameter and $R$ less than 2320, the flow is laminar.

**12.19. Flow of water through a circular pipe.** To illustrate Laws 12.1 and 12.2 we shall study the case of water flow through the horizontal circular pipe of Figure 206. We suppose that the state is steady. The pipe is 10 inches in diameter and 100 feet long with a pressure differential of 1 pound per square foot between the ends.

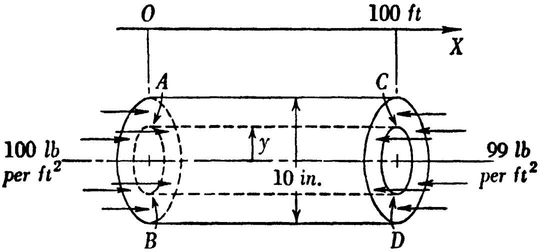


Fig. 206. A pipe containing water.

Since the pressure difference between the ends of this pipe is not great, we would expect a low velocity of the water. We therefore suppose that we have laminar flow.

We introduce an $x$-axis parallel to the pipe axis with $x = 0$ at the left end of the pipe, and the flow of the water toward the right.

Because of symmetry the same state, as far as $u$ is concerned, may be assumed to exist at each point the same distance from the axis of the pipe. For a given distance $y$ all such points lie on a right circular cylinder of radius $y$ with the same axis as the pipe, as shown in Figure 206 for the cylinder $ABDC$.

*Many engineering problems* can be similarly *simplified by making use of symmetry*. We should always look for this.

For every $y$ the body of water within the cylinder is in equilibrium under the action of the forces acting on it. We shall treat the components of these forces in the $x$-direction. The net pressure force on the ends of the cylinder is given by $\pi y^2$. We let $u$ represent the velocity of the points on the lateral surface of the cylinder, this being the same at all such points because the water is taken incompressible, and the continuity Theorem 12.2 applies. Because of symmetry we have $u$ directed toward the right. We suppose that the water is at 68.4 degrees Fahrenheit. By

Law 12.2 we have a shear stress

$$\tau_{yx} = 2.08 \cdot 10^{-5} \frac{du}{dy}. \tag{12.52}$$

Multiplying this stress by the lateral area $200\pi y$ of the cylinder we have a force due to shear given by $0.00416\pi y du/dy$.

By Law 12.1 the velocity of the fluid is zero at the wall, whence the velocity must increase in magnitude as one leaves the wall. Equating to zero the sum of the shear force and the force $\pi y^2$ corresponding to the pressure on the ends, and taking $y \neq 0$ so that we can divide it out of the resulting equation, we have

$$\frac{du}{dy} + 240y = 0. \tag{12.53}$$

We need a boundary condition. This follows from Law 12.1 giving

$$y = \tfrac{5}{12},\ u = 0, \tag{12.54}$$

whence the solution is determined as regards the velocity of the fluid.

We have neglected the effect of gravity in assuming symmetry of the flow with respect to the axis of the pipe. Experience shows that this is justified. It can, of course, be taken into account by a three-dimensional treatment of the problem; this is complicated.

By integrating the solution of (12.53) and (12.54) we can obtain the volumetric rate of flow $Q$ (*Hagen-Poiseuille law*) past any cross section of the pipe. The pressure at any cross section can be obtained by running through the above analysis for a cylindrical element between two arbitrary cross sections, assuming, as is practical, that we can represent the pressure at a section by an average. It is thus proved that the pressure varies linearly from one end of the pipe to the other.

**12.20. Energy in the case of pipe flow.** Bernoulli's theorem does not apply to pipe flow, because the energy density at a point $P$ in the fluid varies, in fact decreases due to friction as the point $P$ moves. In most engineering cases the friction makes itself evident in the form of heat. If we let $H_1$ be the total head for one position of the point $P$, and $H_2$ the total head at a second position, we have

$$H_1 = H_2 + Q \tag{12.55}$$

where $Q$ is heat (density). The quantity $Q$ is called the *loss in head*.

In more general cases we would have to add chemical energy, internal energy, and work terms on the right of (12.55) as explained in Part III.

**12.21. Flow in a trough with an open surface.** We shall analyze the flow of water in the trough of Figure 207 with cross section an isosceles trapezoid of height 2 feet, and bases 4 feet and 8 feet as shown. There is a drop in elevation from the left end of 1 foot for a horizontal distance of 100 feet with the water in steady flow. The trough is assumed full in the sense that the volume of water in it is a maximum.

We introduce an $x$-axis in the direction of the axis of the trough as shown. We may assume that the water has everywhere the same cross section perpendicular to the $x$-axis. We assume also that the flow is steady and laminar. We consider an element of water between the plane sections for $x$ and $(x + \Delta x)$ as shown.

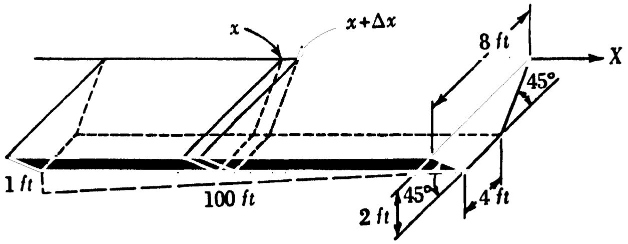


Fig. 207. Flow in a trough with an open surface.

Since the flow is steady and laminar we can neglect the inertia of the water. The component of the force of gravity on the element in the $x$-direction is given by $7.48\Delta x$ pounds. Since the pressure on the surface of the water is everywhere the same, we can expect the pressure to be uniform for practical purposes. We introduce a viscosity force $-cu\Delta x$ for a constant $c$. We assume that $c$ is known. From the equilibrium of forces we have

$$cu = 7.48,$$

whence $u$ is determined.

**12.22. The solution of flow problems.** We have illustrated above some of the principles enumerated below, which should be followed in setting up flow problems in mathematical form. The other principles will be illustrated in chapters to follow.

The problem should be simplified geometrically by making use of symmetry and substitution of geometrically simpler figures whenever possible. The problem should be made one of as few geometric dimensions as possible, preferably one only.

If changes are slow enough, the problem should be treated as one in steady state, thus eliminating initial conditions. If changes are very rapid, inertia should be taken into account, but damping neglected if possible.

If damping is indicated, we should try to use a corresponding force proportional to the velocity of a moving point.

The number of equations should be kept to a minimum by the introduction of a potential function if there is one.

If the flow is one-dimensional we should consider the energy method of deriving an equation of motion. For air flow in the atmosphere, well below the speed of sound, the compressibility of the air should be neglected. If near the speed of sound, compressibility should be considered. If forces on bodies due to flow past these bodies are desired, do not neglect viscosity.

For flow of liquids in conduits, where the changes are not rapid, compressiblity should be neglected but viscosity taken into account. Otherwise, compressibility should be considered. For gases flowing in conduits one should take compressibility into account, but neglect viscosity unless otherwise indicated.

Laminar flow should be assumed when possible.

For flow through orifices it is advisable to neglect friction unless very accurate results are needed.

**12.23. Dimensional analysis applied to fluid problems.** In general the variables that must be considered in fluid motion are the density $\rho$, specific weight $w$, viscosity $\mu$, surface tension $\sigma$, elastic modulus $e$, and a mean speed $V$ (or rate of discharge $Q$) of the fluid, and a pressure increment $\Delta p$ (or gradient $dp/dx$ for a distance coordinate $x$), as well as linear dimensions describing the geometry of the system under consideration. We have length $L$, mass $M$, and time $T$ as primary quantities. Let $l$ be a characteristic linear dimension. By the Buckingham pi theorem we expect five dimensionless groups in the eight secondary quantities whose symbols are given above, namely

$$\frac{\rho V^2}{\Delta p}, \qquad F = \frac{V^2\rho}{wl}, \qquad R = \frac{Vl\rho}{\mu}, \qquad W = \frac{V^2 l\rho}{\sigma}, \qquad C = \frac{V^2\rho}{e}.$$

The numbers $F$, $R$, $W$, and $C$ are the well-known Froude, Reynolds, Weber, and Cauchy numbers. By the Buckingham pi theorem we have the flow equation

$$V^2 = \frac{\Delta p}{\rho} f\left(\frac{l_1}{l}, \frac{l_2}{l}, \ldots, \frac{l_n}{l}, F, R, W, C\right),$$

where the $l$'s are characteristic linear dimensions of the system, and $f$ is a function.

The Reynold's number occurred in Sections 8.16 and 12.18.

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# CHAPTER

# 13

# *Compressible fluids*

**13.1. Water hammer in a circular pipe.** It is well known that a sudden disturbance in the flow of water in a pipe can create noises, a phenomenon known as *water hammer.* It is reasonable to expect that this is due to the compressibility of water, just as *sound waves* are generated in air because of this property. We shall analyze water hammer for circular pipes as an illustration of the manner in which compressibility of liquids is taken into account in engineering problems. The theory applies to any liquid.

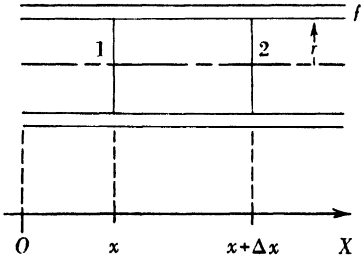

**Fig. 208. Cross section of a circular pipe showing a fluid element.**

It is to be expected that the velocity variations are so small that viscosity can be neglected. We begin with this assumption, and first restrict ourselves to a horizontal pipe.

We suppose that we can work with an average pressure $p$, density $\rho$, and velocity $u$ at each cross section of the pipe. We set up an $x$-axis lengthwise along the pipe as in Figure 208, so that $p$, $\rho$, and $u$ are functions of $x$ and $t$. Consider the fluid section 1 of Figure 208. The coordinate $x$ of a *particle* of fluid at section 1 is a function of $a$ and $t$, where $a$ is the coordinate of the particle when $t = 0$. Since $u$ denotes the velocity of such a particle we have

$$u = \frac{\partial x}{\partial t}$$

for the coordinate $x$ of this particle.

We shall consider the element of fluid bounded by cross sections 1 and 2 of Figure 208. We let $A$ be the cross sectional area at 1, and $\Delta x$ the length of the element.

At the pipe section 1 there is a pressure $p$ on the wall as shown in Figure 209. We now consider a pipe element with the section 1 and length unity, where the pressure is $p$ at each point on the inner surface. This pipe element is not meant to be part of the pipe in Figure 208, but is introduced for the purpose of computation. We shall take $p$ to be gage pressure (relative to the air surrounding the pipe).

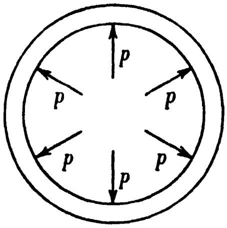

**Fig. 209. Cross section of a pipe.**

We make the reasonable assumption that the pipe thickness $f$ is small compared to the inner radius $r$, and suppose also that the changes in $f$ with time are negligible compared to changes in the inner and outer circumference of the pipe. We pass two planes through the axis of the pipe element cutting out a piece of the pipe element of which a partial cross section is shown in Figure 210. In this figure $T$ represents the magnitude of the tension force on this piece. This means that the piece cut away from the pipe is in equilibrium under the influence of $T$ and the force due to pressure $p$ on the inside of the section. The angle between the cutting planes is denoted by $\Delta\theta$. Because of symmetry there is a force of magnitude $T$ at each side of the piece, acting along the tangent. In analyzing the forces on the piece, we are taking the cross section as an arc of a circle of radius $r$ subtended by $\Delta\theta$. By reason of symmetry, the net force on

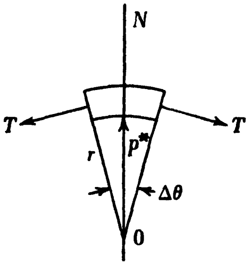

**Fig. 210. Tension force $T$ in a pipe element.**

the piece in the vertical direction $ON$, because of the hydraulic pressure, is of magnitude $p^*r\Delta\theta$ for a number $p^*$. The vertical component of the force represented by $T$ is $T \sin (\Delta\theta/2)$, whence for both sides we have $2T \sin \Delta\theta/2$. Equating to the function representing the net pressure force component in the vertical direction we have

$$p^*r\Delta\theta = 2T \sin \tfrac{1}{2}\Delta\theta. \tag{13.1}$$

Taking the limit in (13.1) as $\Delta\theta$ goes to zero we derive

$$T = pr. \tag{13.2}$$

Let $\sigma_p$ be such that the density of the pipe piece of Figure 210 is $\sigma_p r\Delta\theta$. If we take the inertia of the pipe material into account, the term $T$ in (13.2) must be replaced by $(T + \sigma_p rr'')$ for the time derivative $r''$ of $r$. If $\sigma_p$ is small enough the term $\sigma_p rr''$ is negligible compared to $T$; this we assume. That $\sigma_p rr''$ is negligible in a given problem should be verified from the solution of the water hammer equations.

By *Hooke's law*

$$S = Ee \tag{13.3}$$

for the longitudinal stress $S$ (tension in our case), modulus of elasticity $E$, and longitudinal strain $e$. The *strain* $e$ is the elongation divided by the initial length before $S$ is applied (see Chapter 9).

The area of a face of our piece, to which $T$ is applied, is $f$, whence in this case

$$S = \frac{T}{f}. \tag{13.4}$$

It follows that the strain is $e$ where

$$e = \frac{T}{fE}. \tag{13.5}$$

Let $e_0$ correspond to a value $T_0$ of $T$. By the definition of strain

$$e_0 = \frac{\Delta c_0}{c_0} = \frac{\Delta r_0}{r_0},$$

where $c_0$ is the value of the circumference $c$ of the pipe when $T = 0$, and $r_0$ is the corresponding value of $r$. We write $r = r_0 + \Delta r_0$. If $T$ goes into $(T_0 + \Delta T)$, the strain $e$ goes into $(e_0 + \Delta e)$, where $\Delta e = \Delta T/fE$. Let $\Delta r$ be the corresponding change in $r$. Then $(e_0 + \Delta e)$ is $(\Delta r_0 + \Delta r)/r_0$, whence $\Delta e = \Delta r/r_0$. For any reasonable $r_0$ and $\Delta r_0$ we have

$$\frac{\Delta r}{r_0} \approx \frac{\Delta r}{r_0 + \Delta r_0} = \frac{\Delta r}{r},$$

where the symbol "$\approx$" means "is approximately equal to," as in Section 2.13. Thus, with the *assumption* that $(r_0 + \Delta r_0)$ is the same as $r_0$ for practical purposes, the change in strain $\Delta e$ due to a change $\Delta T$ in $T$ is $\Delta r/r$ for the corresponding change $\Delta r$ in $r$.

We let $\Delta T$ be the change in $T$ over an interval of time $\Delta t$. By (13.2)

$$\Delta T = r\Delta p + p\Delta r + \epsilon \tag{13.6}$$

for a number $\epsilon$. It follows from (13.5) and (13.6) with $\Delta e = \Delta r/r$ that

$$\Delta r = \frac{r^2\Delta p}{(fE - rp)}, \tag{13.7}$$

where we have dropped the $\epsilon$-term because it will not contribute to the final limit. Except for similar $\epsilon$-terms, the change in area $\Delta A$ is $2\pi r\Delta r$ and

$$\Delta A = \frac{2\pi r^3 \dfrac{\partial p}{\partial t}}{(fE - rp)}\Delta t. \tag{13.8}$$

We assume that the deviation from cylindrical shape is small. It follows that *the change in pipe volume* between sections 1 and 2 of Figure 208 is given by $\Delta V_1$ where

$$\Delta V_1 = \Delta A]_3\Delta x \tag{13.9}$$

for $\Delta A$ of (13.8) evaluated at some section 3 between 1 and 2.

**13.2. Change in volume due to compression of a liquid element.** Consider the mass of the fluid between the sections 1 and 2 of Figure 208 at the starting time $t$. During the interval $\Delta t$ the volume occupied by this mass decreases by an amount $\Delta V_2$ due to compression of the fluid.

In a nonviscous liquid the only stress acting is pressure, whence the volume $V$ occupied by a given mass of the liquid is a decreasing function of the pressure. For a change $\Delta p$ in the pressure $p$ there is a corresponding change $\Delta V$ in the volume $V$. For a $\Delta p$ not too large it is reasonable to assume that the relative change in volume is directly proportional to the change in pressure, whence

$$\frac{\Delta V}{V} = -b\Delta p \tag{13.10}$$

for a positive number $b$.

Relation (13.10) yields

$$\Delta p = -K\frac{\Delta V}{V} \tag{13.11}$$

in the ***bulk modulus of elasticity*** $K$. The modulus $K$ is about 300,000 for water when the units are pounds and inches. From numerical values given in Section 12.4 it is seen that the variation of $K$ with temperature in the case of water is not very great.

The *decrease* $\Delta V_2$ *in water volume* is then given by

$$\Delta V_2 = \frac{1}{K}\left[\frac{\partial p}{\partial t}\right]_4 A\Delta x\Delta t, \tag{13.12}$$

where the quantity in the brackets is evaluated at a section 4 between 1 and 2 at some instant between $t$ and $t + \Delta t$, and we have omitted an $\epsilon$-term that does not contribute to the final limit.

We are assuming all continuity properties of functions and their derivatives necessary to carry through the argument.

**13.3. Continuity equation for water hammer.** In the previous section we treated compression of the water. This was done for purposes of exposition only. The same formulas hold for expansion. We shall continue to discuss compression even though the theory also holds for expansion. The net mass rate of flow into the volume section between 1 and 2 of Figure 208 is equal to the mass necessary to fill the volume $(\Delta V_1 + \Delta V_2,$ because of expansion of the pipe wall and compression of the water. We shall therefore compute the net mass flow into the volume element in the interval $\Delta t$. It is convenient to think of going from the state for time $t$ to the state for time $(t + \Delta t)$ by compressing the mass of liquid in the volume element, expanding the pipe to the new dimensions, and filling the space thus vacated with the mass of liquid that has flowed into the element in the time $\Delta t$ minus the mass of liquid which has flowed out in this time. Of course, the net flow in the time $\Delta t$ may be out of the element. The net direction of mass flow is indicated by the ± sign.

Except for an $\epsilon$-term, which can be neglected, the net mass rate of flow into the volume element at the instant $t$ is given by $Q$, where

$$Q = -\frac{\partial \rho A u}{\partial x}\Delta x, \tag{13.13}$$

obtained by subtracting the mass rate of flow out through section 2 from the mass rate of flow in through section 1. The derivative in (13.13) is evaluated at section 1. The net flow during the time $\Delta t$ is

$$Q_5\Delta t = -\left[\frac{\partial \rho A u}{\partial x}\right]_5 \Delta x\Delta t, \tag{13.14}$$

where the subscript 5 indicates that the derivative in (13.14) is evaluated at some time $t_5$ between $t$ and $(t + \Delta t)$.

We shall equate the net flow of (13.14) to the mass of liquid necessary to fill up the volume $(\Delta V_1 + \Delta V_2)$ at the time $(t + \Delta t)$. We let $\rho^*$ be such that this mass is $\rho^*(\Delta V_1 + \Delta V_2)$. The *mass flow relation* is thus

$$Q_5\Delta t = \rho^*(\Delta V_1 + \Delta V_2). \tag{13.15}$$

We make use of (13.8), (13.9), (13.12), and (13.14) in (13.15), divide both sides of equation (13.15) by $\Delta x\, \Delta t$, and take the limit as the maximum of $\Delta t$ and $\Delta x$ go to zero. We are taking $\Delta x$ and $\Delta t$ positive. We obtain

$$\frac{\partial \rho A u}{\partial x} = -\rho A \left[\frac{1}{K} + \frac{2r}{fE - rp}\right] \frac{\partial p}{\partial t}. \tag{13.16}$$

Differentiating $\rho A u$, and using (13.7), we have

$$\frac{\partial \rho A u}{\partial x} = A\left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x}\right) + \frac{2ru\rho A \dfrac{\partial p}{\partial x}}{fE - rp}. \tag{13.17}$$

Hence by (13.16)

$$\left[\frac{\partial u}{\partial x} + \frac{u}{\rho}\frac{\partial \rho}{\partial x}\right] = -\left[\frac{1}{K} + \frac{2r}{fE - rp}\right]\frac{\partial p}{\partial t} - \frac{2ur \dfrac{\partial p}{\partial x}}{fE - rp}. \tag{13.18}$$

In the next section we shall eliminate $\rho$ from (13.18).

## 13.4. Relation between changes in density and pressure.

Let $M$ be a mass of liquid occupying the volume $V$ with density $\rho$. It follows that

$$M = \rho V. \tag{13.19}$$

If we change the pressure $p$ by $\Delta p$, the density changes by $\Delta \rho$ and the volume by $-V\Delta p/K$. Since the mass $M$ remains the same we have

$$\Delta \rho = \frac{(\rho + \Delta \rho)\Delta p}{K}. \tag{13.20}$$

By (13.20)

$$\frac{\partial \rho}{\partial x} = \frac{\rho}{K}\frac{\partial p}{\partial x}. \tag{13.21}$$

Equations (13.18) and (13.21) now yield

$$\frac{\partial u}{\partial x} = -\left[\frac{1}{K} + \frac{2r}{fE - rp}\right]\left[\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right]. \tag{13.22}$$

Equation (13.22) involves the functions $u$ and $p$ of the variables $x$ and $t$. *Equation (13.22) is one of our basic water hammer equations.*

For steel $E$ is about 30,000,000 pounds per square inch; similar large values hold for other pipe materials. It follows that for any practical values of $f$, $E$, $r$, and $p$ the term $rp$ is negligible compared to $fE$, whence we can replace $(fE - rp)$ in (13.22) by $fE$ to get

$$\frac{\partial u}{\partial x} = -\left[\frac{1}{K} + \frac{2r}{fE}\right]\left[\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right]. \tag{13.23}$$

We substitute $h\rho g$ for $p$, the head being $h$, and let $\alpha$ be given by

$$\alpha = \rho g\left[\frac{1}{K} + \frac{2r}{fE}\right]. \tag{13.24}$$

By (13.20) we have

$$dh = \frac{(K - p)dp}{K\rho g}.$$

In practical cases $K - p \approx K$. Then (13.23) can be written as

$$\frac{\partial u}{\partial x} = -\alpha\left[\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right]. \tag{13.25}$$

We need another equation in $u$ and $h$.

**13.5. Force equation for water hammer.** Our second equation in $u$ and $h$ will follow from the dynamic equilibrium of an element of the liquid. We consider a right circular cylindrical element of cross-section area $A$ running from section 1 to section 2 as in Figure 211. Assuming continuity, we can suppose that $\Delta x$ is so short that either the element of liquid of this figure does not meet the wall of the pipe at points between 1 and 2, or such is the case for an element of cross-section area $(A + \Delta A)$, this being the cross-section area of the liquid at 2. The result will be the same in either case. Therefore we work with the element in Figure 211 of base area $A$. The force due to pressure $p$ on the left end of this element is given by $Ap$, and on the right end by $A(p + \Delta p)$, where $(p + \Delta p)$ is the pressure at the right end of the element. The net force due to pressure is given by $A\Delta p$. At the instant $t$ each point in the element has an acceleration in the $x$-direction, which acceleration might be zero. Assuming continuity of $d(\rho u)/dt$ we can find a value $[d(\rho u)/dt]^*$ of $d(\rho u)/dt$ at some point in the element so that the inertial force is represented by $[d(\rho u)/dt]^* A\Delta x$. It follows from Newton's second law that

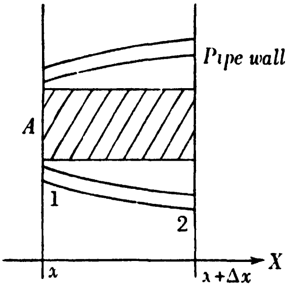

**Fig. 211. Element of fluid.**

$$\left[\frac{d(\rho u)}{dt}\right]^* = -\frac{\Delta p}{\Delta x},$$

whence in the limit as $\Delta x \to 0$

$$\frac{d\rho u}{dt} = -\frac{\partial p}{\partial x}. \tag{13.26}$$

By (13.20) and (13.26) we have

$$\rho\frac{du}{dt} = -\left[\frac{\rho u}{K}\frac{dp}{dt} + \frac{\partial p}{\partial x}\right].$$

Since $\rho u/K$ is probably very small compared to 1, we write this equation as

$$\rho\frac{du}{dt} = -\frac{\partial p}{\partial x}.$$

Now $u$ is a function of $x$ and $t$. Therefore,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}.$$

Since $p = \rho gh$, relation (13.21) yields

$$\frac{\partial p}{\partial x} = \frac{g\rho\frac{\partial h}{\partial x}}{\left[1 - \frac{h\rho g}{K}\right]}.$$

With $K$ large we have

$$\frac{\partial p}{\partial x} = g\rho\frac{\partial h}{\partial x}.$$

The relation (13.26) has now been reduced to

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x}. \tag{13.27}$$

We can expect $\partial u/\partial t$ and $\partial h/\partial t$ to be generally large compared to $u\partial u/\partial x$ and $u\partial h/\partial x$, respectively. We therefore can simplify the relations (13.25) and (13.27) to

$$\frac{\partial u}{\partial x} = -\alpha\frac{\partial h}{\partial t}, \qquad \frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x}. \tag{13.28}$$

Under our simplifying assumptions we have thus arrived at the *partial differential equations (13.28) of water hammer* in the pressure head $h$, velocity $u$, distance $x$, and time $t$. With boundary conditions these equations suffice to describe the motion of the liquid through the pipe when viscosity is neglected.

In our treatment we have disregarded stresses and strains in the pipe that are concerned with the axial direction of the pipe. This is justified by experience.

**13.6. Equations for a slanted pipe.** Suppose that the pipe of Section 13.1 makes an angle $\theta$ with the horizontal. For an element of liquid between sections 1 and 2 as in Figure 212 there is an additional force due to gravity. Take the element as in Figure 211. In addition to the applied force represented by $A\Delta p$ there is the gravity force given by

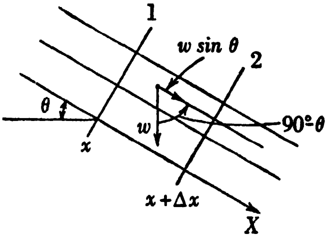

Fig. 212. Slanted pipe.

$\rho g A \Delta x \sin \theta$. In (13.26) we have the extra term $\rho g \sin \theta$ on the right. It follows that the second equation in (13.28) goes into

$$\frac{\partial u}{\partial t} = g\left(\sin \theta - \frac{\partial h}{\partial x}\right). \tag{13.29}$$

The first equation in (13.28) and equation (13.29) cover the slanted pipe case.

**13.7. Allievi's equations.** The *Allievi equations* in (13.28) may be combined to give the wave equation

$$\frac{\partial^2 h}{\partial x^2} = \frac{\alpha}{g} \frac{\partial^2 h}{\partial t^2}.$$

See Section 5.2 for a discussion of the wave equation. For $\alpha$ Allievi used $g/a^2$ because $a$ is the speed of the pressure wave; in fact we have the solution

$$h = F(at - x) + f(at + x)$$

for functions $F$ and $f$.

**13.8. Friction in pipe flow.** There are cases in which friction in the liquid cannot be neglected. It is then logical to assume that the friction force is proportional to a power $u^n$ of the speed $u$. For laminar flow, experiments show that $n = 1$, and for turbulent flow $n$ ranges from 1.7 for a smooth pipe to 2 for a rough pipe. Taking the friction force proportional to $u$, the equations of motions are modified by the replacement of (13.29) by

$$\frac{\partial u}{\partial t} = g\left(\sin \theta - ku - \frac{\partial h}{\partial x}\right) \tag{13.30}$$

for a constant $k$.

*To analyze the water hammer problem we have made assumptions indicated by the mathematics itself to simplify the mathematics. This is typical of engineering studies.*

The difficulties we have encountered in treating the problem of compressibility for liquid flow in a pipe are typical of compressibility problems in general, and show why the subject, so important for high speed aircraft, is so little understood.

**13.9. Sudden closing of a pipe.** We shall analyze the case of the horizontal circular steel pipe of Figure 213, where the pipe is 200 feet long with diameter 1 foot and thickness $\frac{1}{2}$ inch. The pressure at the pipe inlet is 100 pounds per square inch gage and the pressure at the outlet is atmospheric when this end is open. The speed of the water is 10 feet per second in steady state until the right end of the pipe is instantly closed. It is supposed that we are working at normal temperatures.

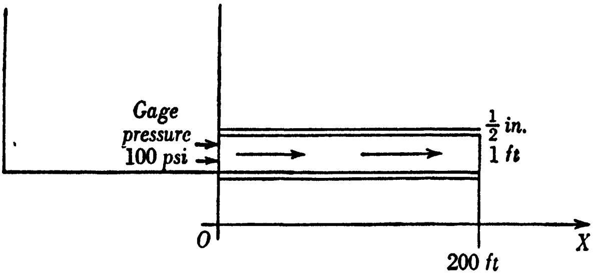

Fig. 213. Pipe suddenly closed.

We neglect friction, and let $t = 0$ be the instant of closure. We measure our variables from the steady state assumed to exist before the closure. With $x$ chosen as in the figure, the boundary conditions for $x$ and $h$ in inches are

$$x = 0, \quad h \overset{t}{\equiv} 0; \qquad x = 2400, \quad u \overset{t}{\equiv} -120 \tag{13.31}$$

for identities in $t$. With pounds, inches, and seconds we have the constants

$$r = 6, E = 28{,}000{,}000, K = 300{,}000, g = 386, \rho g = 0.0362.$$

Our differential equations (13.28) are

$$\frac{\partial u}{\partial x} = -123 \times 10^{-9} \frac{\partial h}{\partial t}, \qquad \frac{\partial u}{\partial t} = -386 \frac{\partial h}{\partial x}. \tag{13.32}$$

If the pipe is slanted 30 degrees with the horizontal the second equation in (13.32) has the extra term 193 on the right.

We have not made use of the pressure 100 pounds per square inch at the intake because we measured our variables from the steady state. The absolute pressures vary over such a limited range that the density $\rho$ in $\alpha$ was taken constant.

**13.10. Sudden opening of a pipe.** If the pipe of Figure 213 is closed at the right end at the start with a steady state existing throughout the pipe, and if we suddenly open the right end we have the boundary conditions

$$x = 0, \quad h = 0; \qquad x = 2400, \quad h = -2{,}760,$$

holding identically in $t$ during the disturbance. The equations (13.32) still hold.

**13.11. Pipe with different sections.** We shall analyze the case of Figure 213 with the pipe replaced by the two-piece arrangement of Figure 214 where a 50 foot pipe 2 feet in diameter is joined to a 150 foot pipe 1 foot in diameter.

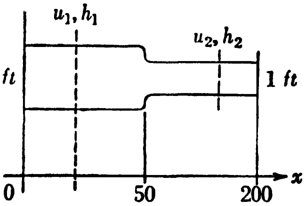

**Fig. 214. Pipe with different sections.**

We let $u_1$ and $h_1$ denote the velocity and pressure head at a section of the left piece, and $u_2$, $h_2$ the corresponding variables for a section of the right piece. The equations (13.28) apply with $u$ and $h$ replaced by $u_1$ and $h_1$ at the left piece, and similarly for the right piece with $u$ and $h$ replaced by $u_2$ and $h_2$. At the junction we have the conditions

$$x = 50, \; u_2 = 4u_1, \; h_2 = h_1$$

holding for each instant. The relation between the $u$'s comes from the continuity of flow across the junction. For the disturbance of Section 13.9 the same boundary conditions hold. For the disturbance of Section 13.10 the boundary conditions again hold.

**13.12. Gases.** Gases are treated in some detail in Part III devoted to heat. We shall therefore discuss only two examples here in which the compressibility of gases is taken into account. These are given in the next section and Section 13.14 where we shall treat rapidly changing phenomena for which the adiabatic law

$$pV^{\gamma} = C \tag{13.33}$$

will be needed; here $p$ is the pressure of the gas, $V$ is the volume or specific volume (volume per unit weight), and $\gamma$ and $C$ are constants.

**13.13. Piston in a cylinder.** Let us analyze the problem of a piston weighing two pounds moving in the cylinder of Figure 215. The interior cross section of the cylinder has an area of 10 square inches. The coordinate of the piston is denoted by $x$ with $x = 0$ at the base of the cylinder. The absolute pressure in the cylinder is designated by $p$. We shall treat the problem where the piston is initially at rest at $x = 4$ when the pressure is 500 pounds per square inch, the gas in the cylinder is air, and the air outside of the cylinder is about normal atmospheric temperature. The piston is suddenly freed to move under the influence of the net pressure of air inside the cylinder and of the atmosphere outside.

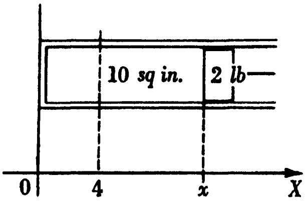

Fig. 215. Piston in a cylinder.

Since we are working with air the adiabatic exponent $\gamma$ in (13.33) is approximately 1.4. Since $V = 40$ when $p = 500$, it follows that $C = 500 \cdot 40^{1.4}$. We assume that there is no leakage of the air past the piston.

The force acting on the piston is represented by $10(p - 14.7)$. Friction is neglected. We assume that at each instant $t$ the pressure $p$ is everywhere the same in the cylinder. Since $V = 10x$, we have

$$p = 500\left(\frac{4}{x}\right)^{1.4} \tag{13.34}$$

The mass of the piston is $\frac{1}{193}$. It follows from Newton's second law that

$$x'' = 1{,}930\left(\frac{3{,}500}{x^{1.4}} - 14.7\right). \tag{13.35}$$

Relation (13.35) is a nonlinear second-order differential equation. Our initial conditions are

$$t = 0,\ x = 4,\ x' = 0. \tag{13.36}$$

**13.14. Sound.** Let us consider the general problem of sound in a gas. The local motion of the gas is normally small although rapid. If gravity and viscosity forces are neglected equations (12.4) apply with $X = Y = Z = 0$, assuming that the density changes are small.

Simple considerations of geometric volume elements show that the continuity equation (12.7) of Theorem 12.2 becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{13.37}$$

for the *general fluid case.* In vector form (13.37) is

$$\operatorname{div} \rho V + \frac{\partial \rho}{\partial t} = 0,$$

where $V$ is the velocity of the fluid. Equations (12.4) and (13.37) are equations in the variables $x$, $y$, $z$, $p$, and $\rho$. We need another equation to determine these variables. Since the motions of sound are rapid it is to be expected that these take place without heat changes, that is, *adiabatically.* By Chapter 6 we may suppose that the relation (13.33) holds. We write this as

$$p = c\rho^{\gamma} \tag{13.38}$$

for constants $\gamma$ and $c$.

To simplify the system of equations it is convenient to write

$$\rho = \rho_0(1 + \sigma) \tag{13.39}$$

for an *average* density $\rho_0$. It is assumed that $\sigma$ is small compared to 1. Substitute formula (13.39) for $\rho$ in (13.37), divide both sides of the resulting equation by $\rho_0$, and replace $(1 + \sigma)u$ by $u$, and $(1 + \sigma)v$ and $(1 + \sigma)w$ by $v$ and $w$, respectively. The last step is justified because of the smallness of the motion. We have now

$$\frac{\partial \sigma}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{13.40}$$

in place of (13.37).

By (13.38), (13.39), and the binomial theorem

$$p = c\rho_0^{\gamma}(1 + \sigma)^{\gamma} = c\rho_0^{\gamma}\left[1 + \gamma\sigma + \frac{\gamma(\gamma - 1)}{2!}\sigma^2 + \cdots\right]. \tag{13.41}$$

We use the last formula of (13.41) for $p$, neglecting terms of the second degree and higher in $\sigma$. By (12.4)

$$\frac{\partial u}{\partial t} = -k^2\frac{\partial \sigma}{\partial x}, \qquad \frac{\partial v}{\partial t} = -k^2\frac{\partial \sigma}{\partial y}, \qquad \frac{\partial w}{\partial t} = -k^2\frac{\partial \sigma}{\partial z}, \tag{13.42}$$

where

$$k^2 = \frac{c\gamma\rho_0^{\gamma-1}}{1 + \sigma}.$$

We write 1 for $1 + \sigma$ in $k^2$ so that $k^2$ becomes a constant. By (13.40) and (13.42) we have

$$\frac{\partial^2 \sigma}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2} + \frac{\partial^2 \sigma}{\partial z^2} = \frac{1}{k^2}\frac{\partial^2 \sigma}{\partial t^2}. \tag{13.43}$$

Equation (13.43) is the standard equation of sound wave motion. Its solution gives a wave with speed $k$. If $p_0$ is the average air pressure, for which $p_0 = c\rho_0^\gamma$, we thus obtain the usual form

$$k = \sqrt{\frac{p_0\gamma}{\rho_0}}$$

of $k$. For standard air $\gamma = 1.4$ as noted in Section 13.13.

The well-known properties of sound waves follow from the differential equations of motion derived in this section.

## REFERENCE

1. Ernest W. Schoder and Francis M. Dawson, *Hydraulics*, McGraw-Hill, New York, 1934.

## CHAPTER

## 14

# *Aerodynamics*

**14.1. Flow past an airplane wing.** We shall consider the problem of steady air flow past the wing of an airplane with an *average* cross section as in Figure 216. Since the distance from wing tip to wing tip is long compared with other wing dimensions, we replace the wing by a cylinder with cross section perpendicular to its axis identical with an *average* cross section of the wing. We thus reduce our problem to a two-dimensional one.

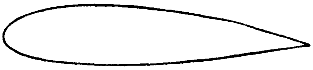

**Fig. 216. Cross section of an airplane wing.**

To treat two-dimensional flow it is convenient to employ the concepts of velocity potential and circulation introduced in the next section.

**14.2. Velocity potential and circulation.** Multiply the equations in (12.4) by $\partial x/\partial a$, $\partial y/\partial a$, $\partial z/\partial a$, respectively, and add. Thus

$$\left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial a} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial a} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial a} + \frac{1}{\rho}\frac{\partial p}{\partial a} = 0.$$

We can obtain from (12.4) two similar equations in partial derivatives with respect to $b$ and $c$. We suppose that the force intensities represented by $X$, $Y$, and $Z$ have a potential function $\Omega$ (we have shown in this book that we can associate a potential function with a large class of forces in nature), so that

$$X = -\frac{\partial \Omega}{\partial x}, \; Y = -\frac{\partial \Omega}{\partial y}, \; Z = -\frac{\partial \Omega}{\partial z}.$$

We now have

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial \Omega}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a}, \tag{14.1}$$

and there are two similar equations in partial derivatives of $x$, $y$, $z$, $p$ and $\Omega$ with respect to $b$ and $c$, respectively.

We suppose that the derivatives and the density $\rho$ that occur in these equations are continuous functions for each set of values $(a, b, c, t)$ under consideration. It is convenient to suppose that the points $(a, b, c, t)$ belong to a simply connected region $R_1$ of the 4-space with the Cartesian coordinates $a$, $b$, $c$, and $t$. We assume that $\rho$ is a function of $p$ only, and does not vanish. We shall integrate the equations of the type (14.1) along a streamline (see Section 12.9).

The law for the derivative of a product yields

$$d(UV) = UdV + VdU$$

from which

$$UdV = d(UV) - VdU,$$

and

$$\int_0^t UdV = UV \Big]_0^t - \int_0^t VdU$$

where the bracket indicates that we are to take $UV$ evaluated at $t = t$ minus $UV$ evaluated at $t = 0$.

Assume that $\partial^2x/\partial a\partial t$ exists and is continuous from 0 to $t$. Integrating by parts we have

$$\int_0^t \frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a}\,dt = \frac{\partial x}{\partial t}\frac{\partial x}{\partial a}\Big]_0^t - \int_0^t \frac{\partial x}{\partial t}\frac{\partial^2 x}{\partial a\partial t}\,dt.$$

Since $\partial x/\partial a = 1$ at $t = 0$,

$$\int_0^t \frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a}\,dt = \frac{\partial x}{\partial t}\frac{\partial x}{\partial a} - u_0 - \int_0^t \frac{\partial x}{\partial t}\frac{\partial^2 x}{\partial a\partial t}\,dt$$

for the initial value $u_0$ of $u$. We let $p_0$ be a value of $p$; for convenience let this be the value of $p$ at $t = 0$. The integral

$$\int_{p_0}^{p} \frac{dp}{\rho}$$

is assumed to exist, and is a function $f(p)$ of $p$. Now

$$\frac{\partial f(p)}{\partial a} = \frac{df(p)}{dp}\frac{\partial p}{\partial a} = \frac{1}{\rho}\frac{\partial p}{\partial a}.$$

Now

$$\frac{\partial}{\partial a}\int_0^t \left[\int_{p_0}^{p} \frac{dp}{\rho}\right] dt = \int_0^t \frac{1}{\rho}\frac{\partial p}{\partial a}\,dt.$$

Since the second-order partial derivatives of $x$, $y$, and $z$ with respect to $t$ exist, the first derivatives of $x$, $y$, and $z$ with respect to $t$ exist and are continuous functions of $t$. Suppose that $\Omega$ and $p$ are continuous functions of $t$. Since $f(p)$ is a continuous function of $p$, and $p$ is a continuous function of $t$, the function $f(p)$ is continuous with respect to $t$. It follows that the function $\psi$ given by

$$\psi = \int_0^t \left[ \int_{p_0}^{p} \frac{dp}{\rho} + \Omega - \frac{1}{2}\left\{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2\right\} \right] dt$$

exists. We suppose that $\partial x/\partial t$, $\partial^2 x/\partial t \partial a$, $\Omega$, and $\partial \Omega/\partial a$ are continuous over $R_1$. Integrating both sides of the equations of type (14.1) from 0 to $t$, we have:

$$\begin{aligned}
\frac{\partial x}{\partial t}\frac{\partial x}{\partial a} + \frac{\partial y}{\partial t}\frac{\partial y}{\partial a} + \frac{\partial z}{\partial t}\frac{\partial z}{\partial a} - u_0 &= -\frac{\partial \psi}{\partial a}, \\
\frac{\partial x}{\partial t}\frac{\partial x}{\partial b} + \frac{\partial y}{\partial t}\frac{\partial y}{\partial b} + \frac{\partial z}{\partial t}\frac{\partial z}{\partial b} - v_0 &= -\frac{\partial \psi}{\partial b}, \\
\frac{\partial x}{\partial t}\frac{\partial x}{\partial c} + \frac{\partial y}{\partial t}\frac{\partial y}{\partial c} + \frac{\partial z}{\partial t}\frac{\partial z}{\partial c} - w_0 &= -\frac{\partial \psi}{\partial c}.
\end{aligned} \tag{14.2}$$

With the continuity assumptions made above we have the following theorem due to Lagrange. For its statement we need the concept of *velocity potential.* It is by definition a function $\phi$ for which

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z},$$

provided $u$, $v$, and $w$ are such that $\phi$ exists.

***Theorem* 14.1.** *Let a velocity potential exist at a given instant for the portion P of a nonviscous fluid in a simply connected region R where the fluid is in motion under action of forces which have a potential. Let the density of the fluid be a function of the pressure only (it might be a constant). If now the continuity assumptions used to derive equations (14.2) are satisfied, a velocity potential exists at each instant for the portion P of the fluid.*

At the instant $t = 0$ the relation

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

takes the form

$$-d\phi_0 = u_0 da + v_0 db + w_0 dc$$

since $(x, y, z) = (a, b, c)$ at $t = 0$. Multiplying the equations in (14.2) by $da$, $db$, and $dc$, respectively, and adding, we find that

$$\frac{\partial x}{\partial t} dx + \frac{\partial y}{\partial t} dy + \frac{\partial z}{\partial t} dz = -d(\phi_0 + \psi)$$

where $t$ is constant in the differentials $dx$, $dy$, and $dz$. This can be written as

$$u dx + v dy + w dz = -d\phi$$

for $\phi$ given by

$$\phi = \phi_0 + \psi.$$

It follows that $\phi$ is the desired potential function valid at the instant $t$ under consideration.

The potential function $\phi$ applies to the portion $P$ of the fluid as long as this portion occupies a simply connected part $R^*$ of space such as the region $R$ at $t = 0$. Then there is a one-to-one correspondence between the points of $R$ and $R^*$, where two corresponding points can be joined by a streamline. If the region $R^*$ is no longer simply connected, the theory breaks down. Thus if the portion $P$ meets the wing of an airplane and envelops this wing, there are points where $P$ comes together after the wing and these points correspond to two points of $R$.

At a considerable distance from the wing of an airplane the velocity field is considered uniform, and therefore has a velocity potential. Thus if $v$ and $w$ vanish, and the component of velocity denoted by $u$ is a constant, we have the potential function $\phi$, where $\phi = -ux$. Consider the wing to be an infinitely long cylinder of constant cross section, and let air flow past the cylinder be in steady state. For every region not containing the wing there is then a velocity potential.

For any kind of flow the line integral

$$\int_C udx + vdy + wdz \tag{14.3}$$

along a curve $C$ joining points $A$ and $B$ is by definition the *flow* from $A$ to $B$. If $C$ is a closed curve, this integral is said to be the *circulation* $\Gamma$ around the curve $C$. If $V$ is the velocity at a point on $C$, and $\theta$ is an angle between the direction of $V$ and a tangent ray to the circle at this point, the integral (14.3) becomes

$$\int_C |V| \cos \theta ds, \tag{14.4}$$

which in vector form is simply

$$\int_C \mathbf{V} \cdot d\mathbf{s}.$$

Assuming that Stokes's theorem applies, we have

$$\int_C \mathbf{V} \cdot d\mathbf{s} = \iint_\Sigma \textbf{curl } \mathbf{V} \cdot d\mathbf{\Sigma}$$

for a simply closed curve $C$ and a surface $\Sigma$ bounded by $C$. The vanishing of $\Gamma$ for the closed curves in a region $R$ implies that

$$\textbf{curl } \mathbf{V} = 0$$

for this region. When there is a velocity potential $\phi$ for $R$ we have

$$\Gamma = \int_C \mathbf{V} \cdot d\mathbf{r} = -\int_C d\phi = 0.$$

*These considerations apply to any fluid, not only to air.*

Returning to the case of the infinitely long wing, the circulation $\Gamma$ around any simply closed curve not enclosing the wing vanishes, provided of course the integral defining $\Gamma$ exists. We shall show that the circulation $\Gamma$ around any such curve enclosing the wing taken counterclockwise is a constant independent of the curve (see Section 5.1 for an analogous treatment in the case of a wire carrying an electric current). Consider a simply closed curve $C_2$ enclosing the wing as in Figure 217. We let $C_1$ be a simply closed curve lying on the surface of the wing, and we join a point $A$ on $C_2$ to a point $B$ on $C_1$ by means of a curve $C_3$ that does not otherwise intersect $C_1$ and $C_2$. We integrate from

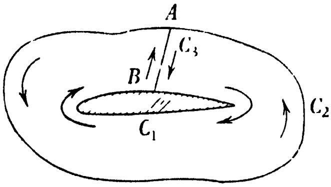

**Fig. 217. Curve $C_2$ around an airfoil.**

the point $A$ around $C_2$ counterclockwise back to the point $A$, then along the curve $C_3$ to $B$, followed by the integral around $C_1$ back to $B$ clockwise, and then along $C_3$ to $A$, the integral being of the form (14.4), where $C$ is the composite curve made up of the components $C_2$, $AB$, $C_1$, and $BA$. The curve $C$ bounds a simply connected region, whence the integral (14.4) vanishes. Since the integrals from $A$ to $B$ and $B$ to $A$ cancel each other, the vanishing of (14.4) implies that

$$\int_{C_2} |V| \cos \theta ds + \int_{C_1} |V| \cos \theta ds = 0,$$

which means that

$$\int_{C_2} |V| \cos \theta ds = \int_{C_1} |V| \cos \theta ds,$$

where the last integral is now taken around $C_1$ counterclockwise. Thus the circulation is a property of the flow past the wing as a whole.

**14.3. Lift and drag.** We consider infinitely long wings of constant cross section where the wings are a common distance $h$ apart as in Figure 218. There are infinitely many such identical wings parallel to each other with the leading edge lying on a vertical line $AB$, and the trailing edge on another vertical line not shown in the figure. In what follows the velocities are taken with respect to an $xy$-coordinate system (not shown in the figure) attached to a wing. We suppose that the same state exists at each wing, and that we have steady laminar flow past the wings. A *particle* of fluid then describes a streamline $CN$ between two wings as shown, it being supposed that there is no component of velocity of the fluid in the direction of the generators of a wing; that is, the flow is two-dimensional. On the other side of one of these wings there is a corresponding streamline *FE* located in the same way as *CN* between two consecutive wings. The points *C* and *F* are chosen so that the straight-line segment *CF* is vertical. Similarly the straight-line segment *NE* is vertical. The line segment *CF* is taken ahead of a wing, and *NE* behind the wing. Consider the curve *CNEF* formed by the streamline *CN*, the line segment *NE*, the streamline *EF*, and the line segment *FC*. We shall compute the circulation Γ around *CNEF* by integrating in the counterclockwise direction.

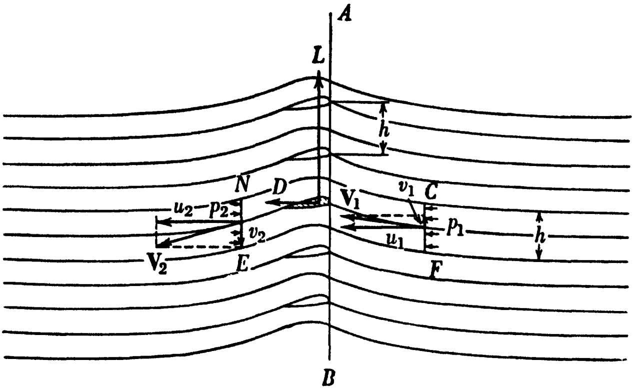

**Fig. 218. A row of identical airfoils.**

If *CF* is far enough away from the wing, and this line segment is long enough, we would expect the velocity to be reasonably uniform for the points of the line segment *CF*. We let $V_1$ be the velocity at the points on the line segment *CF*, and let $u_1$ and $v_1$ represent the horizontal and vertical components of $V_1$. In the same manner we associate with the line segment *NE* a velocity $V_2$ and components given by $u_2$ and $v_2$. The integral (14.4) from *F* to *C* along *FC* is now $v_1h$, and from *N* to *E* along *NE* this integral is $-v_2h$. The line integral (14.4) from *E* to *F* along the streamline *EF* is the negative of this line integral from *C* to *N* along the streamline *CN*. It follows that the circulation Γ around *CNEF* is given by

$$\Gamma = h(v_1 - v_2). \tag{14.5}$$

We associate with the line segments *CF* and *NE* average pressures, or at least representative pressures $p_1$ and $p_2$, respectively, these pressures being normal to these line segments. The force acting on a wing has the vertical component *L*, which is the *lift* on the airfoil, and a horizontal component with numerical value *D*, termed the *drag*. It is convenient to consider the airfoil sections between two planes $\pi_1$ and $\pi_2$ a unit distance apart and perpendicular to the generators of these airfoils. Suppose that the plane of the paper in Figure 218 is one of these planes, say $\pi_1$. Corresponding to the line segment *CF* there is a line segment *GH* in the other plane $\pi_2$, so that *CF* and *GH* are sides of a rectangle *CFHG* with the other two sides parallel to generators of the wings. There is a similar rectangle *NEJM*, where *JM* is a line segment of the plane $\pi_2$. These rectangles and the streamlines through the horizontal line segments *CG* and *FH* bound a region *R* enveloping a wing section between the planes $\pi_1$ and $\pi_2$. We let $\rho$ be the density of the air, assumed to be uniform in the present argument. The mass rate at which fluid crosses the face *CFHG* of *R* is $dm/dt$, where

$$\frac{dm}{dt} = \rho h u_1.$$

During the interval $(t, t + \Delta t)$ a mass $\rho h u_1 \Delta t$ of fluid crosses the face *CFHG*. Since no fluid crosses the streamlines, the mass rate at which fluid crosses the face *NEJM* is $dm/dt$, but this is $\rho h u_2$. It follows that

$$u_1 = u_2.$$

In the interval $(t, t + \Delta t)$ a mass $\rho h u_1 \Delta t$ of fluid crosses the face *NEJM*, and leaves the region *R*. At the instant $(t + \Delta t)$ this mass of fluid has an ***average*** vertical component of velocity $v_2^*$, so that the vertical component of the momentum of this mass at this instant is given by $\rho h u_1 v_2^* \Delta t$. Of the mass $m_0$ of fluid in the region *R* at the instant *t*. a part is left in this region at the instant $(t + \Delta t)$, at least if $\Delta t$ is small enough. Let the vertical component of momentum of this part at the instant $(t + \Delta t)$ be denoted by $M_0$. At the instant $(t + \Delta t)$ the mass $m_0$ of fluid thus has the vertical component of momentum given by

$$M_0 + \rho h u_1 v_2^* \Delta t.$$

At the instant *t* the mass $m_0$ of the fluid had a vertical component of momentum given by

$$M_0 + \rho h u_1 v_1^* \Delta t,$$

where $v_1^*$ is an average value of the vertical component of velocity for the fluid of mass $\rho h u_1 \Delta t$ near the face *CFHG*. Thus during the time $\Delta t$ the vertical component of momentum of the mass of fluid in the region *R* at the instant *t* changes by an amount represented by $\Delta M$, where

$$\Delta M = \rho h u_1 (v_2^* - v_1^*) \Delta t.$$

It follows that the instantaneous rate of change $dM/dt$ corresponding to the vertical component of momentum of the fluid in the region *R* at the instant *t* is

$$\frac{dM}{dt} = \rho h u_1 (v_2 - v_1). \tag{14.6}$$

By Newton's second law of motion there is a vertical component of force acting on the fluid in the region *R*. Let this component be represented by

$F_v$. Then

$$F_v = \frac{dM}{dt}.$$

Since the fluid applies a lift $L$ to the wing, the wing applies an equal and opposite force, represented by $-L$, to the fluid. We shall neglect the influence of gravity on the fluid, assuming that $L$ is large compared to the weight of the fluid in the region $R$, or we suppose that our physical system is not under the influence of a gravitational field. Because the streamlines $CN$ and $FE$ are identical in shape, and identically placed in the fluid, the forces on the top face of the region $R$, if there are any, are balanced by equal and opposite forces on the bottom face of $R$. The forces acting on the ends $CFHG$ and $NEJM$ are due to hydrostatic pressure on these ends. Since this is normal to the ends we have no vertical component of force acting on these ends. It follows that $F_v$ is $-L$. By (14.6)

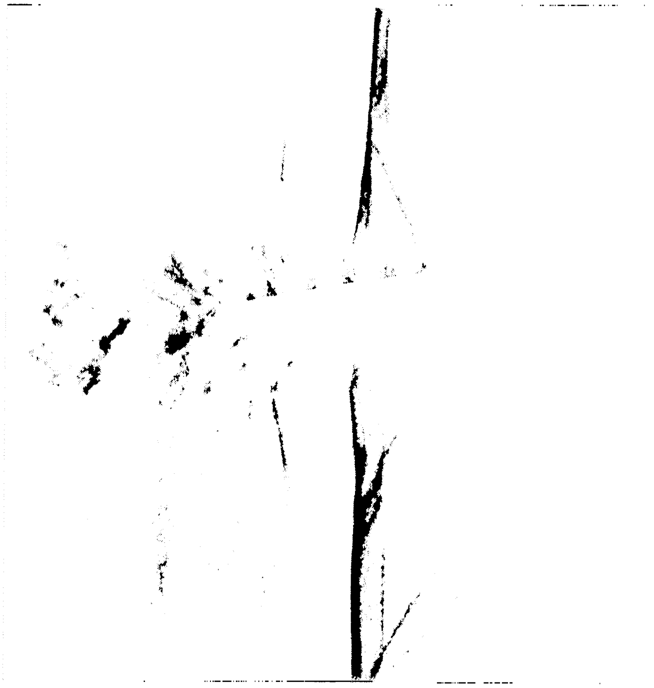

**Fig. 219. Schlieren photograph of flow past an airfoil showing a shock wave and wake formation. Made in the transonic wind tunnel of the Guggenheim Aeronautical Laboratory. Courtesy of the California Institute of Technology.**

$$L = \rho h u_1(v_1 - v_2). \tag{14.7}$$

By (14.5) and (14.7)

$$L = \rho \Gamma u_1. \tag{14.8}$$

For the horizontal component of the rate of change of momentum of the fluid in the region $R$ we have (14.6) with $v_2$ and $v_1$ replaced by $u_2$ and $u_1$, respectively. Since $u_1 = u_2$ this component of the rate of change of momentum vanishes. The horizontal forces applied to the fluid in $R$ thus add up to zero. Due to pressure on the ends of the region $R$ we have the horizontal component of force represented by $h(p_2 - p_1)$. The force represented by $D$ is applied by the wing to the fluid. It follows that

$$D = h(p_1 - p_2).$$

By Bernoulli's theorem

$$p_1 + \frac{\rho|V_1|^2}{2} = p_2 + \frac{\rho|V_2|^2}{2}.$$

Since $|V_1|^2 = u_1^2 + v_1^2$, $|V_2|^2 = u_2^2 + v_2^2$, and $u_1 = u_2$, this gives

$$p_1 - p_2 = \tfrac{1}{2}\rho(v_2^2 - v_1^2).$$

Thus

$$D = \tfrac{1}{2}\rho h(v_2^2 - v_1^2).$$

In terms of the circulation $\Gamma$ we can write $D$ as

$$D = -\tfrac{1}{2}\rho\Gamma(v_1 + v_2). \tag{14.9}$$

Let $\delta$ be the minimum of the quantities $h$, the distance from $AB$ to $CF$, and the distance from $AB$ to $NE$. Keeping the wing in the region $CNEF$ fixed, and increasing $\delta$ beyond all bounds while the circulation $\Gamma$ remains constant, the limit of $(v_1 - v_2)$ must be zero. Actually, the limits of $v_1$ and $v_2$ are both zero if the velocity of the fluid approaches perpendicularity to the line $AB$ as we recede from $AB$, which is assumed. Let the limiting velocity of the fluid relative to the wing be given by $V_0$. By (14.8) and (14.9) we have in the limit

$$L = \rho\Gamma V_0, \; D = 0. \tag{14.10}$$

We arrive at the famous *Kutta-Joukowski theorem* which states that *the lift L per unit length for an infinitely long cylindrical wing, with circulation $\Gamma$ around it, is given by (14.10), where $\rho$ is the density of the fluid, and $V_0$ is the limiting speed of the fluid as we recede from the wing.*

For an actual wing the drag is not zero due to skin friction and other factors.

*The treatment above applies to any homogeneous, incompressible, nonviscous fluid*, that is, any ideal fluid.

The lift results from suction on top of the wing, and an increase in pressure underneath.

The pressure $p_s$ attained when a fluid is brought to rest, as when it strikes a wall head on, is called the *stagnation pressure.* Let $p_0$ and $V_0$ be the values of the pressure and speed at some instant before contact with the wall. By Bernoulli's theorem

$$p_s - p_0 = \tfrac{1}{2}\rho V_0^2.$$

The pressure rise $(p_s - p_0)$ due to bringing the fluid to rest is called the *dynamic pressure.*

In practice the lift $L$ is written as

$$L = C_L(\tfrac{1}{2}\rho V_0^2)S,$$

and the drag $D$ as

$$D = C_D(\tfrac{1}{2}\rho V_0^2)S.$$

Here $C_L$ and $C_D$ are the lift and drag coefficients of the wing respectively, and $S$ is a characteristic area. The lift and drag are thus expressed in terms of the dynamic pressure.

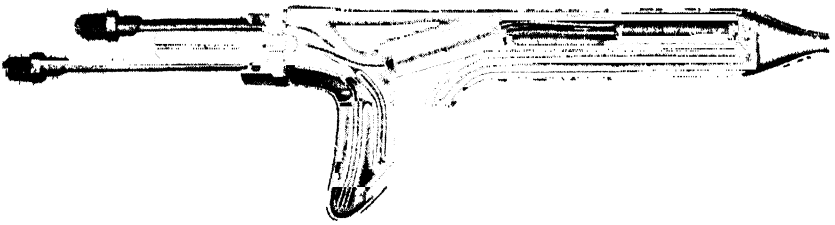

**Fig. 220. Pitot-static tube for measuring air speed. Courtesy of the Kollsman Instrument Division of the Square D Company.**

Air speed is commonly measured by a *pitot-static tube* which records dynamic pressure. See Figure 220 for a photograph of such a tube.

## REFERENCES

1. Hans Wolfgang Liepmann and Allen E. Puckett, *Introduction to Aerodynamics of a Compressible Fluid*, Wiley, New York, 1947.
2. M. J. Zucrow, *Principles of Jet Propulsion and Gas Turbines*, Wiley, New York, 1948.
3. Clark B. Millikan, *Aerodynamics of the Airplane*, Wiley, New York, 1941.
4. L. Prandtl and O. G. Tietjens, *Fundamentals of Hydro- and Aeromechanics*, and *Applied Hydro- and Aeromechanics*, McGraw-Hill, New York, 1934.

# Problems for Part V

The reader is to express the following flow problems in mathematical form, and is not expected to solve them.

1. The plate of Figure 203 is inclined 45° to the jet of water. We are interested in the force on the plate.
2. The plate of Figure 203 is inclined 85° to the jet of water. The spring is strong enough to keep the plate moving horizontally. The plate has an initial velocity of 10 in per sec at $x = 0$. The plate weighs $\frac{1}{2}$ pound.
3. The bucket of a waterwheel is curved. We are interested in the force on the bucket produced by a jet on it when the wheel is turning.
4. We are interested in $z$ for Figure 201 when friction and leakage at the piston are taken into account.
5. Introduce a small hole with area $\frac{1}{2}$ sq ft, 10 ft up from the bottom of the reservoir of Figure 200.
6. A ship rocks on a calm ocean.
7. Take leakage and friction into account in the piston problem of Section 13.13.
8. A vertical cylindrical vessel containing sulfuric acid is rotating about its axis with constant acceleration. We are interested in the surface of the acid.
9. The vessel of Problem 8 is accelerated at a constant rate from zero velocity. We are interested in the pressure.
10. Water is flowing steadily from a large reservoir through a pipe to a junction where the pipe splits into two smaller ones that discharge into the atmosphere. Suddenly the open end of one of these smaller pipes is closed.
11. In the problem of Section 13.9 the pipe end is closed as to area, linearly with time, instead of instantly.
12. Hydrogen in a rubber balloon is at constant temperature. The balloon is suddenly released, the temperature in the balloon remaining at that of the atmosphere.
13. The balloon of Problem 12 is floating in the air. Hydrogen is escaping through a constant orifice.
14. A pump is bringing oil from a low pressure to a closed high-pressure tank. There is leakage past the pump. We are interested in the oil flow.
15. Helium is flowing from a tank at constant high pressure through a pipe to atmospheric pressure. We are interested in the flow when the atmospheric end is suddenly closed.
16. In Problem 15 the pipe is closed and the helium is at rest. Suddenly the pipe is opened.

**17.** There is a sinusoidal variation of the pressure on the surface of oil in a high-pressure tank from which the oil flows through a pipe into the air.

**18.** One end of a pipe containing air is closed. At the other end of the pipe water pours in under constant pressure. We are interested in the state of the air.

**19.** Two tanks are connected by a horizontal pipe joining the bottoms of the tanks. One contains water, and the other is empty. The tops of the tanks are open to the air. A valve in the horizontal pipe, which is first closed, is opened and its effective area as an orifice then varies sinusoidally with time. We are interested in the flow.

**20.** A vertical cylinder is filled with air, confined at the bottom of the cylinder by a spring-loaded piston. Oil suddenly rushes into the cylinder at the bottom. The oil is under constant supply pressure. There is leakage past the piston. We are interested in the motion of the piston.

**21.** The force holding a submarine on the bottom of the sea is suddenly removed and it rises to the surface.

**22.** A body heavier than water drops to the bottom of the sea.

**23.** A bubble of air rises in water.

**24.** Air flows steadily past an elliptical cylinder.

**25.** A right circular cylindrical column of water rests upon a plane horizontal surface. The cylinder is instantly removed and the water spreads over the surface.

**26.** An upright tube of small cross section is filled with oil. We are interested in the shape of the surface of the oil.

**27.** A right circular cylinder is moved at very high speed through the air, the direction of motion being perpendicular to the axis of the cylinder.

**28.** By dimensional analysis treat laminar flow through a circular pipe.

**29.** Treat the flow of the preceding problem by dimensional analysis under the assumption that the flow is turbulent instead of laminar.

**30.** By dimensional analysis find a formula for the drag on an airplane.

**31.** By dimensional analysis find a formula for the speed of a wave in deep water.

**32.** Find a formula for the speed of sound in a gas. Use dimensional analysis.

**33.** By dimensional analysis determine the effect of the density and viscosity of a fluid on the speed $V$ of efflux from an orifice, given that $V$ varies as the square root of the pressure head across the orifice.

# APPENDIX

# *Dimensions of common quantities*

Dimensions of mechanical quantities in terms of mass $M$, length $L$, and time $T$.

| | $M$ | $L$ | $T$ |
|---|---|---|---|
| Acceleration | 0 | 1 | −2 |
| Angle | 0 | 0 | 0 |
| Angular acceleration | 0 | 0 | −2 |
| Angular speed | 0 | 0 | −1 |
| Area | 0 | 2 | 0 |
| Density | 1 | −3 | 0 |
| Elastic modulus | 1 | −1 | −2 |
| Force | 1 | 1 | −2 |
| Frequency | 0 | 0 | −1 |
| Momentum | 1 | 1 | −1 |
| Power | 1 | 2 | −3 |
| Strain | 0 | 0 | 0 |
| Stress, pressure | 1 | −1 | −2 |
| Torque | 1 | 2 | −2 |
| Velocity | 0 | 1 | −1 |
| Viscosity | 1 | −1 | −1 |
| Volume | 0 | 3 | 0 |
| Work, energy | 1 | 2 | −2 |

Dimensions of electrical quantities in terms of mass $M$, length $L$, time $T$, current $I$, resistance $R$, permeability $\mu$, and permittivity $\epsilon$.

| | SYSTEM | | | | SYSTEM | | | | SYSTEM | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $M$ | $L$ | $T$ | $\epsilon$ | $M$ | $L$ | $T$ | $\mu$ | $R$ | $I$ | $L$ | $T$ |
| Capacitance | 0 | 1 | 0 | 1 | 0 | −1 | 2 | −1 | −1 | 0 | 0 | 1 |
| Charge | $\frac{1}{2}$ | $\frac{3}{2}$ | −1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 1 | 0 | 1 |
| Charge density (volumetric) | $\frac{1}{2}$ | $-\frac{3}{2}$ | −1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{5}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 1 | −3 | 1 |
| Conductance | 0 | 1 | −1 | 1 | 0 | −1 | 1 | −1 | −1 | 0 | 0 | 0 |
| Conductivity (electrical) | 0 | 0 | −1 | 1 | 0 | −2 | 1 | −1 | −1 | 0 | −1 | 0 |
| Current | $\frac{1}{2}$ | $\frac{3}{2}$ | −2 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | −1 | $-\frac{1}{2}$ | 0 | 1 | 0 | 0 |
| Current density | $\frac{1}{2}$ | $-\frac{1}{2}$ | −2 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{3}{2}$ | −1 | $-\frac{1}{2}$ | 0 | 1 | −2 | 0 |
| Elastance | 0 | −1 | 0 | −1 | 0 | 1 | −2 | 1 | 1 | 0 | 0 | −1 |
| Electric displacement (electric flux density) | $\frac{1}{2}$ | $-\frac{1}{2}$ | −1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{3}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 1 | −2 | 1 |
| Electric field intensity | $\frac{1}{2}$ | $-\frac{1}{2}$ | −1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | −2 | $\frac{1}{2}$ | 1 | 1 | −1 | 0 |
| Electric flux | $\frac{1}{2}$ | $\frac{3}{2}$ | −1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 1 | 0 | 1 |
| Energy | 1 | 2 | −2 | 0 | 1 | 2 | −2 | 0 | 1 | 2 | 0 | 1 |
| Force | 1 | 1 | −2 | 0 | 1 | 1 | −2 | 0 | 1 | 2 | −1 | 1 |
| Inductance | 0 | −1 | 2 | −1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| Intensity of magnetization | $\frac{1}{2}$ | $-\frac{3}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | −1 | 1 | $\frac{1}{2}$ | 1 | 1 | −2 |
| Magnetic field intensity | $\frac{1}{2}$ | $\frac{1}{2}$ | −2 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | −1 | $-\frac{1}{2}$ | 0 | 1 | −1 | 0 |
| Magnetic flux | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | −1 | $\frac{1}{2}$ | 1 | 1 | 0 | 1 |
| Magnetic flux density | $\frac{1}{2}$ | $-\frac{3}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | −1 | $\frac{1}{2}$ | 1 | 1 | −2 | 1 |
| Magnetic pole strength | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | −1 | $\frac{1}{2}$ | 1 | 1 | 0 | 1 |
| Magnetomotive force | $\frac{1}{2}$ | $\frac{3}{2}$ | −2 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | −1 | $-\frac{1}{2}$ | 0 | 1 | 0 | 0 |
| Permeability | 0 | −2 | 2 | −1 | 0 | 0 | 0 | 1 | 1 | 0 | −1 | 1 |
| Permittivity | 0 | 0 | 0 | 1 | 0 | −2 | 2 | −1 | −1 | 0 | −1 | 1 |
| Potential difference | $\frac{1}{2}$ | $\frac{1}{2}$ | −1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | −2 | $\frac{1}{2}$ | 1 | 1 | 0 | 0 |
| Power | 1 | 2 | −3 | 0 | 1 | 2 | −3 | 0 | 1 | 2 | 0 | 0 |
| Reluctance | 0 | 1 | −2 | 1 | 0 | −1 | 0 | −1 | −1 | 0 | 0 | −1 |
| Resistance | 0 | −1 | 1 | −1 | 0 | 1 | −1 | 1 | 1 | 0 | 0 | 0 |

Dimensions of thermal quantities in terms of temperature $\theta$, mass $M$, length $L$, time $T$, and heat $H$.

| | $\theta$ | $M$ | $L$ | $T$ | $H$ |
|---|---|---|---|---|---|
| Coefficient of heat transfer | −1 | 0 | −2 | −1 | 1 |
| Coefficient of thermal expansion of a fluid | −1 | 0 | 0 | 0 | 0 |
| Enthalpy | 0 | 0 | 0 | 0 | 1 |
| Entropy | −1 | 0 | 0 | 0 | 1 |
| Heat capacity (specific heat) per unit mass | −1 | −1 | 0 | 0 | 1 |
| Heat capacity per unit volume | −1 | 0 | −3 | 0 | 1 |
| Heat capacity per unit weight | −1 | −1 | −1 | 2 | 1 |
| Thermal conductivity | −1 | 0 | −1 | −1 | 1 |

# ANSWERS AND HINTS

## PART I

1. Let $x$ be the coordinate of position of the body. A solution is

$$x'' + 386x = 0;\ t = 0,\ x = 1,\ x' = 0.$$

2. For the coordinate $x$ of the plunger we have

$$x' + 2x = 0;\ t = 0,\ x = 2,\ x' = 0.$$

3. Let $x'$ denote the velocity of the sphere relative to the water. Suppose that the sphere has a zone of height $x$ in contact with the water. Assume that the resistance of the water is a function $-g(x, x')$. Let $\rho$ be the density of the sphere, and $V$ its volume. The equation of motion is then

$$\rho V x'' + g(x, x') = 0.$$

If the speed of the sphere is not too high, we may take $g(x, x')$ to be a linear combination of $x$ and $x'$, so that the equation of motion becomes

$$\rho V x'' + x(k_1 x + k_2 x') = 0.$$

4. Let $x$ be the coordinate of the body, and $m$ its mass. We can then write

$$mx'' + kx = 0;\ t = 0,\ x = 0,\ x' = x_0'.$$

5. Let $x$ be the coordinate of the higher body measured as a deviation from equilibrium, and let $m_1$ be its mass. Let $y$ and $m_2$ be the corresponding quantities for the lower body. We shall use $k_1$ and $k_2$ for the scales of the higher and lower springs respectively. The positive directions for both $x$ and $y$ are taken downward. The free body equations are now

$$m_2 y'' + k_2 y = k_2 x,$$
$$m_1 x'' + (k_1 + k_2)x = k_2 y.$$

6. Let $m$ be the mass of the body, and $x$ its coordinate measured from equilibrium, with the downward direction as positive. Let $k$ be the scale of the spring. For the first downward stroke we can have

$$mx'' + kx = 0;\ t = 0,\ x = 0,\ x' = x_0'.$$

Let $l$ be such that $x = l$ is the coordinate of the base. When the body strikes the base it does so with a velocity represented by $x_1'$. For the rebound stroke use the same differential equation, but the initial conditions

$$t = 0,\ x = l,\ x' = -x_1'.$$

7. The bumper is like a spring. When the car is in contact with the bumper we have an equation of the type

$$mx'' + cx' + kx = 0,$$

and when not in contact, an equation of the form

$$mx'' + cx' = 0.$$

8. Oil causes damping, and an equation of the type

$$mx'' + cx' + kx = 0$$

results.

9. We have a system of the type

$$IN' + f(N) = 0; t = 0, N = N_0,$$

in the speed $N$ of the propeller.

10. Let $m$ be the mass of the object, $w$ its weight, and $(x, y)$ its coordinates. We can describe the motion of the object by the system

$$\begin{aligned} mx'' + \cos\theta\, f(\sqrt{(x')^2 + (y')^2}) &= 0, \\ my'' + \sin\theta\, f(\sqrt{(x')^2 + (y')^2}) + w &= 0. \\ t = 0,\ v_x = v_{x0},\ v_y = 0;\ \tan\theta &= \frac{y'}{x'}. \end{aligned}$$

11. Replace the crankshaft by a set of right circular cylindrical disks joined by elastic rod members. The pistons and connecting rods are replaced by equivalent inertias. See the book by Den Hartog referred to at the end of Chapter 2. Let $\theta_x$ be the angular displacement of the $x$th disk. Let $\theta_f$ be the angular displacement of the flywheel. A system of the form

$$\begin{aligned} I\theta_n'' &= c(\theta_{n-1} - \theta_n), \\ I\theta_x'' &= c(\theta_{x-1} - \theta_x) - c(\theta_x - \theta_{x+1}), \qquad 1 < x < n, \\ I\theta_1'' &= c(\theta_2 - \theta_1) - k_0(\theta_1 - \theta_f), \\ I_0\theta_f'' &= k_0(\theta_1 - \theta_f) \end{aligned}$$

is valid.

12. For the mass $m$ of the boat and coordinate $x$ there is an equation

$$mx'' + c(x')^n = T,$$

where $T$ is the propeller thrust.

13. An equation of the form

$$IN' = A \sin \omega t$$

holds.

14. Suppose we have a friction load. In this case an equation of the type

$$IN' + cN = 0$$

holds.

15. For the moment of inertia $I$ of the body and angular coordinate $\theta$ of the body, the energy equation takes the form

$$\tfrac{1}{2}I(\theta')^2 + \tfrac{1}{2}k\theta^2 = K.$$

**16.** Due to wheel friction an equation

$$mx'' + cx' = 0$$

holds.

**17.** Due to pressure on the elevators there is a force $F$ acting perpendicularly to the elevators and making the acute angle $\alpha$ with the vertical. Suppose that the elevators are elevated. Let $d$ be the moment arm of $F$ with respect to the center of gravity of the airplane. The system of equations (2.90) will be modified to take care of the present problem. Take the magnitude of $F$ proportional to $\alpha$ as well as the speed $v$ of the airplane. The equations (2.90) are now replaced by

$$\begin{aligned} I\phi'' &= -M + k\alpha vd, \\ mx'' &= -L \sin \theta - D \cos \theta - k\alpha v \sin \alpha, \\ my'' &= L \cos \theta - D \sin \theta - k\alpha v \cos \alpha - w. \end{aligned}$$

**18.** Let $x$ be the distance the saw has bitten into the wood, and let $f(x)$ be the corresponding torque on the rotating member. An equation

$$IN' = f(x)$$

holds for the angular speed $N$ of the saw.

**19.** Let the angular displacement of the beam from equilibrium be designated by $\theta$, and let $W$ denote the given weight. Let $d$ be the moment arm from the weight to the fulcrum. An equation of the form

$$I\theta'' + c\theta' = Wd$$

holds. Initial conditions may be $t = 0$, $\theta = 0$, $\theta' = 0$.

**20.** The entire spring-pulley system can be replaced by an equivalent spring with scale $x$. For the coordinate $x$ of the weight, we can have

$$mx'' + kx = 0.$$

**21.** An equation

$$mx'' + cx' + kx = F$$

holds for a constant $F$.

**22.** Let $W$ be the weight of the shovel and dirt. For a horizontal coordinate $u$ of position of the shovel there is a load torque proportional to $Wu$ on the engine. The throttle torque is $kx$ for the throttle coordinate $x$. The speed $N$ of the crankshaft of the engine now satisfies an equation

$$IN' + cN + KWu = kx.$$

There is an additional equation

$$x' = c.$$

**23.** Suppose that the bodies move along an $x$-axis, and let $x_1$ be the $x$-coordinate of one of the bodies with mass $m_1$, while $x_2$, $m_2$ are the corresponding quantities for the other body. Let $k$ be the spring scale. Assume that after motion starts there is a dry friction force $K_1$ on the body with mass $m_1$, and a corresponding force $K_2$ on the other body. Let $d$ be the normal length of the spring. Now

$$\begin{aligned} m_1x_1'' &= k\,[x_2 - x_1 - d] \pm K_1, \\ m_2x_2'' &= -k\,[x_2 - x_1 - d] \pm K_2. \end{aligned}$$

The signs are such that the friction forces oppose the motion. Also

$$t = 0,\ x_1 = x_{10},\ x_2 = x_{20},\ x_1' = x_2' = 0.$$

**24.** Let $I_1$ be the engine moment of inertia, $\theta_1$ the angular displacement of the crankshaft, $n\theta$ the angular displacement of the driving gear, $\theta$ the angular displacement of the driven gear, $\theta_2$ the angular displacement of the generator rotor, and $k_1$, $k_2$ torsional stiffnesses of shafts. The equations of motion of the engine shaft and the generator rotor are

$$I_1\theta_1'' + k_1(\theta_1 - n\theta) = 0,$$
$$I_2\theta_2'' + k_2(\theta_2 - \theta) = 0.$$

Since forces at the gears are balanced ($1:n$ gear ratio) the equation

$$\frac{k_1(n\theta - \theta_1)}{r} = \frac{k_2(\theta_2 - \theta)}{nr}$$

is valid. Here $r$ is the radius of the driving gear.

**25.** Let $m$ be the mass of the body and study the motion of its center of gravity $(x, y)$ only. The origin of the $xy$-system is taken at the center of the disk, and the surface of the disk is in the $xy$-plane. Suppose that the body starts to move immediately. Let $\theta$ be an angle which the velocity vector of the body at the center of gravity makes with the positive half of the $x$-axis. Then for a constant $K$

$$mx'' = -K\cos\theta,$$
$$my'' = -K\sin\theta;$$
$$t = 0,\ x = x_0,\ y = y_0,\ x' = y' = 0.$$

**26.** Let $x$ be the piston coordinate, so that

$$mx'' = kt^2.$$

**27.** Assume that once a lath leaves one spoke it flips instantly to the next. Suppose that there is a constant friction $K$. As the lath bends, a spring force $-k\theta$ is applied to the spoke, where $\theta$ is a properly measured angle of rotation of the wheel. During contact with one spoke the equation

$$IN' = -k\theta - K$$

is valid for the speed $N$ of the wheel.

**28.** See Section 2.6. Treat all blocks together as one spring and dashpot. An equation of motion of the type

$$m(x + y)'' + cx' + kx = 0$$

applies where $y$ is a function of $t$ like $A \sin \omega t$.

**29.** Consider a portion of the disk of mass $\Delta m$ at a *point* $P$ with vertical coordinate $y$ measured from the equilibrium position of the axle, and a distance $r$ from the center $C$ of rotation. Let $y_0$ be the $y$-coordinate of the center of rotation, and introduce $\bar{y}$, so that $y = y_0 + \bar{y}$. Let $(\theta + \omega t)$ be an angle between $CP$ and the upward vertical direction, where $\omega$ is the angular velocity of the disk. It is convenient to think of $\omega t$ as an angle between the upward vertical and a fixed ray in the disk, and $\theta$ an angle between this ray and $CP$. Now

$$\bar{y}'' = -r\omega^2 \cos(\omega t + \theta),$$

so that

$$y''\Delta m = y_0''\Delta m - r\omega^2 \cos(\omega t + \theta)\Delta m.$$

Let $k$ be the scale of the spring. The negative of the vertical component of the inertial force is

$$\iint_{\text{disk}} y''dm$$

and is equal to the spring force $-ky_0$, whence

$$My_0'' - \omega^2 \iint_{\text{disk}} r\cos(\omega t + \theta)\rho d\Sigma = -ky_0$$

for the mass $M$ of the disk and density $\rho$.

**30.** Let $m$ be the mass of the body, $r$ the distance from the center of the circle to the center of gravity of the body, and $\omega$ the angular speed. The centrifugal force has the magnitude $mr\omega^2$, and is directed outward.

**31.** Let $x$ and $N$ be the throttle coordinate and engine rpm, respectively, measured as deviations from equilibrium. The governor equation is $x = k_1N$, and the engine equation $IN' = k_2x$.

**32.** See references to Synge and Griffith, and von Kármán and Biot at the ends of Chapters 1 and 2.

**33.** As secondary quantities use the time $t$ of the swing, and amplitude $\theta$ of the swing, the length $l$ of the pendulum, the mass $m$ of the bob, and the magnitude $g$ of the acceleration of gravity. We note that $\theta$ is dimensionless. There is one dimensionless product $tl^{-\frac{1}{2}}g^{\frac{1}{2}}$ with $t$ to the first power, whence

$$t = \sqrt{\frac{l}{g}}\, f(0)$$

for a function $f$.

**34.** Let the sinusoidal force be given by $F\cos\omega t$. The secondary quantities are $m$, $k$, $c$, $F$, $\omega$, and the amplitude $x_M$. Three dimensionless groups are

$$\pi_1 = \frac{kx_M}{F}, \qquad \pi_2 = \omega\sqrt{\frac{m}{k}}, \qquad \pi_3 = \frac{c}{\sqrt{mk}},$$

and the solution is $\pi_1 = f(\pi_2, \pi_3)$ for a function $f$.

**35.** There is one dimensionless $\pi$-term $vg^{-1}t^{-1}$ with the speed $v$ to the first power. This term is equal to a constant.

## PART II

**1.** Let $E$ be the battery emf, $i$ the current in the circuit, $R$, $C$, and $L$ the resistance, capacitance, and inductance, respectively. At $t = 0$ there is a drop $E$ across the capacitor. Thus

$$Li' + Ri + \frac{1}{C}\left[CE + \int_0^t i dt\right] = 0.$$

Also

$$t = 0,\ i = 0.$$

2. Replace the capacitance by a resistance in parallel with a capacitance.
3. See the M.I.T. book on *Magnetic Circuits and Transformers* referred to at the end of Chapter 4.
4. Let $i_1$ be the current through the capacitor, and $i_2$ the current through the load resistance. Let the input alternating voltage be $E \sin \omega t$, and let $R$ designate the resistance of the diode during conduction. Let $R_L$ be the load resistance, and introduce $L$ and $C$ for the indicated inductance and capacitance. During the conduction period

$$L\frac{di}{dt} + Ri + \frac{1}{C}\left[CE_0 + \int_0^t i_1 dt\right] = E \sin \omega t,$$
$$R_L i_2 = E_0 + \frac{1}{C}\int_0^t i_1 dt,$$
$$i = i_1 + i_2.$$

During the nonconduction period the diode resistance should be taken very large, or infinite.

5. Let $C$ and $L$ denote the capacitance and inductance, respectively, $R_C$ the resistance adjoining the capacitance, and $R_L$ the resistance adjoining the inductance. Let $i_1$ be the current through the branch containing $C$ and $L$, and $i_2$ the current through the other branch. Then

$$Li' + \frac{1}{C}\left[CE_0 + \int_0^t i_1 dt\right] = (R_L + R_C)i_2 = e_i,$$
$$e_0 = R_L i_2 - Li_1'.$$

6. Write $e = f(t)$. Let $i$ be the current in the coil, and $x$ the plunger position. The force on the plunger due to the current is of the form $F(i, x)$. Suppose that the plunger is in such a position that $F(i, x) = Ai + Bx$ for numbers $A$ and $B$. Our differential equations are of the form

$$Li' + Ri = f(t),$$
$$mx'' + kx = Ai + Bx.$$

7. See Section 4.14, from which the equations

$$\frac{\partial e}{\partial x} = L\frac{\partial i}{\partial t} + Ri, \qquad \frac{\partial i}{\partial x} = C\frac{\partial e}{\partial t} + Ge$$

are valid for the current $i$ and voltage $e$ at a point of the line. If the load resistance is $R_L$, and $x$ denotes distance along the line with $x = 0$ at the generator and $x = a$ at the load, we have the boundary conditions

$$x = 0,\ e = 200\sqrt{2} \sin \omega t,$$
$$x = a,\ e = R_L i.$$

8. Let $m$ be the mass of the electron, $\boldsymbol{v}$ its velocity, $q$ its charge, and $\boldsymbol{B}$ the magnetic induction. Then $m\boldsymbol{a} = q\boldsymbol{v} \times \boldsymbol{B}$.
9. Suppose that the electric intensity $\boldsymbol{E}$ is constant. With the coordinate axes

set up properly with respect to the plates, assumed to be vertical, and $E$ represented by a function $E(x)$, the equations of motion are

$$\begin{aligned} mx'' &= Eq, \\ z'' &= -g. \end{aligned}$$

10. Let the charge be $Q$. The electric field intensity $\boldsymbol{E}$ at a point a distance $r$ from the center of the sphere to a point outside the sphere satisfies the relation

$$|\boldsymbol{E}| = \frac{|Q|}{kr}$$

for a constant $k$. See the reference to Atwood at the end of Chapter 3.

11. See the M.I.T. book on applied electronics referred to at the end of Chapter 4.
12. See the reference above.
13. The battery supplies an electric motor. Let $N$ be the speed of the crankshaft. Let $i$ be the current in the motor, and $E$ the supply voltage. Equations of the form

$$\begin{aligned} IN' + cN &= ki^2, \\ Li' + Ri &= E \end{aligned}$$

hold if we suppose that the motor is a series motor so that the field flux density is proportional to $i$. Thus since the current in the rotor windings is $i$, there will be a torque proportional to $i^2$ on the rotor.

14. Let $\theta$ be the numerically smallest angle between the equilibrium position of the plate and the instantaneous position. We suppose that the magnetic pull on the plate is $k_2\theta$ for a constant $k_2$. We then have a system

$$\begin{gathered} I\theta'' + k_1\theta = k_2\theta, \\ t = 0,\ \theta = \theta_0,\ \theta' = \theta_0'. \end{gathered}$$

15. Let $e$ be the voltage supplying the motor, and $i$ the current through the motor windings. Let $N$ be the speed of the rotor. There will then be a counter emf, which we shall take proportional to $N$. We write $e = ct$. Then an equation

$$Li' + Ri = ct - k_1N$$

is valid. Suppose that the friction is a constant $K$ during the time considered. A torque equation of the form

$$IN' + cN + K = k_2i^2$$

holds. It may be possible to use an average value of $i$ and write $k_3i$ in place of $k_2i^2$.

16. Let the given angle be denoted by $\theta$, and let $f(\theta)$ be the torque on the needle due to the field. There is then a torque equation of the form

$$I\theta'' + c\theta = f(\theta).$$

Under proper conditions $f(\theta)$ may be linear in $\theta$.

17. Write $R = At + B$. In the usual notations we have

$$Li' + (At + B)i + \frac{1}{C}\left[CE_0 + \int^t i\,dt\right] = E$$

for the current $i$.

**18.** Let the battery emf be designated by $E$, and the other elements as usual. Let $i$ be the current through the capacitor. Then

$$Li' + Ri + \frac{1}{C}\left[CE + \int_0^t i\,dt\right] = 0;$$
$$t = 0,\ i = 0.$$

**19.** Treat the opened switch as if there were a capacitor of small capacitance in the circuit.

**20.** In the usual notations

$$\frac{1}{C}\left[CE_0 + \int_0^t i\,dt\right] + Ri = E,$$
$$e = Ri,$$
$$C = C_0 \sin \omega t$$

for the current $i$ in the closed circuit shown in the figure.

**21.** Let $R_0$ be the initial value of the resistance, and $R$ the final value. Let $E$, $L$, $R$, and $i$ be defined as usual. Then

$$Li' + Ri = E,$$
$$t = 0,\ i = \frac{E}{R_0}.$$

**22.** We have simply the case where a differential equation

$$L_1 i' + Ri = E$$

is suddenly replaced by another of the form

$$L_2 i' + Ri = E.$$

The current is invariant during the inductance change.

**23.** Let $E$ be the battery emf, and $E_0$ the potential difference across the capacitor at the instant the discharge begins. Let $i_1$ be the current through the capacitor, $i_2$ the arc current, and let the voltage drop $e$ across the arc be given by $e = K - ki_2$ for constants $K$ and $k$. Then equations of the form

$$E = R(i_1 + i_2) + \frac{1}{C}\left[CE_0 + \int_0^t i_1\,dt\right],$$
$$\frac{1}{C}\left[CE_0 + \int_0^t i_1\,dt\right] = K - ki_2$$

hold.

**24.** With the proper notations we have the system of equations

$$L_1 i' + \frac{1}{C_1}\left[C_1E_1 + \int_0^t i\,dt\right] + Li_1' = e_i,$$
$$Li_1' = \frac{1}{C_2}\left[C_2E_2 + \int_0^t i_2\,dt\right],$$
$$Li_1' = \frac{1}{C_3}\left[C_3E_3 + \int_0^t i_3\,dt\right] + L_3i_3' + e_0,$$
$$i = i_1 + i_2 + i_3.$$

25. See the M.I.T. book on electric circuits referred to at the end of Chapter 3.
26. Let $\theta$ be the angular deviation of the generator rotor from synchronism with the line current. There is then a synchronizing torque $k\theta$. A differential equation of the type
$$I\theta'' + c\theta' + k\theta = 0$$
now holds.
27. Let $\psi$ be the light intensity, and $R$ the resistance of the cell. Then $R = R_0 - k\psi$. Let $i$ be the current in the solenoid. An equation of the type
$$i = \frac{\mu E}{R}$$
holds. We also have an equation
$$mx'' + cx' + kx = Ai + Bx$$
for the coordinate $x$ of the plunger.
28. The secondary quantities are $F$, $i$, $m$, and $d$. As primary quantities use length $L$, mass $M$, time $T$, and permeability $\mu$. For the dimensions of the secondary quantities see the Appendix. There is a dimensionless product $Fi^{-1}m^{-1}d$. Setting this equal to a constant we obtain a formula which agrees in form with equation (3.16).
29. Let $V$ be the speed of the wave and use $R^*$, $L^*$, and $C$ as in Section 4.28. The secondary quantities are $V$, $R^*$, $L^*$, and $C$. Take resistance $R$, time $T$, and length $L$ as primary quantities. The dimensions of $V$, $R^*$, $L^*$, and $C$ are $LT^{-1}$, $RL^{-1}$, $RTL^{-1}$, and $R^{-1}TL^{-1}$ respectively. The product $V(L^*)^{\frac{1}{2}}C^{\frac{1}{2}}$ is dimensionless. Setting this equal to a constant $k$, we have
$$V = \frac{k}{\sqrt{L^*C}}.$$
Actually, experiment shows that this formula holds with $k = 1$ if $R^*$ is small.
30. The secondary quantities are $i$, $t$, $k$, $r$, and $e_0$. Take potential difference $E$, current $I$, and time $T$ as primary quantities. By (3.46) the dimensionality of $\psi$ is $ET$. The dimensionality of $k$ is $I(ET)^{-7}$, and that of $r$ is $EI^{-1}$. By the Buckingham pi theorem we need two dimensionless quantities. We can take them to be $ir/e_0$ and $(kre_0^6)^{\frac{1}{7}}t$, whence
$$i = \frac{e_0}{r} f\,[(kre_0^6)^{\frac{1}{7}}t]$$
for a function $f$.

## PART III

1. Assume that the temperatures $T_a$, $T_c$, and $T_0$ of the trapped air, container, and outside air (or other medium) respectively are uniform. Get an equation of the form
$$w_a c_p V_a \frac{dT_a}{dt} = h_a A_a (T_c - T_a)$$

from the fact that the rate at which heat is stored in the trapped air equals the rate at which this air receives heat by convection and radiation from the wall. We have used here a combined convection-radiation coefficient $h_a$. Since the rate at which heat is stored in the wall equals the rate at which heat by convection and radiation is received from the outside minus the rate at which heat is given off by convection and radiation to the trapped air and by radiation to the outside, the equation

$$w_c c_{pc} V_c \frac{dT_c}{dt} = h_c A_c (T_0 - T_c) - h_a A_a (T_c - T_a)$$

holds, where we again have used a combined convection-radiation coefficient $h_c$ for the heat transfer between the container and the outside.

2. Let $T$ be the absolute temperature at a cross section of the rod. At the top end no heat is conducted away from this end, whence

$$\frac{\partial T}{\partial x} = 0, \; x = L$$

is a boundary condition. The differential equation (8.31) applies.

3. The plate receives heat by convection (and radiation) from the bottom, and except for storage, gives this off above. Assume uniform temperature in the plate.
4. Assuming that friction can be neglected, the rod loses heat (or gains heat) by convection (and radiation). The coefficient of heat transfer increases with the speed. Suppose that this varies directly as the speed. Assume that at each instant the temperature throughout the rod is uniform.
5. This problem is the same as 4 above, except that the coefficient of heat transfer is constant.
6. Assume that the rate of heat generated by skin friction is proportional to the square of the speed, and also proportional to the density of the air. Take storage in the skin into account. Assume a uniform temperature of the plane inside of the skin, and take into consideration the heat transfer from the skin to the interior of the plane by convection (and radiation), as well as from the outside air to the plane.
7. Assume that the heat generated is proportional to the square of the speed and also proportional to the density of the air.
8. Work with enthalpies of the air and fuel. Assume that Bernoulli's theorem applies.
9. Heat energy at the rate $Ri^2$ is given to the iron by the resistors. Part of this is stored, and the rest convected and radiated to the air.
10. Since the bulb is evacuated, neglect heat transfer by convection from the filament. Neglect also conduction along the filament to the base of the bulb. Consider radiation from the filament to the glass, absorption, transmission, and reflection at the glass, storage of heat in the glass, and heat transfer by convection and radiation to the atmosphere. Assume that the temperature in the filament is uniform, and that the temperature in the glass is also uniform.
11. Neglect the change in volume of the balloon. The rate of heat lost from the stored gas equals the rate transferred from the gas to the casing by convection

(and radiation). Take storage in the casing into account as well as heat transfer by convection (and radiation) to the atmosphere.

12. The heat of vaporization is supplied by coils and in turn from the medium around them. See the reference to Zemansky at the end of Chapter 6.
13. Assume that the rate of heat generated by friction and going into the wheel is constant. Consider the temperature gradient in the wheel, storage of heat in the wheel, and convection (and radiation) of heat to the air. Suppose that the temperature is the same at all points the same distance from the center of the wheel.
14. Storage, conduction, convection, and radiation of the containers and liquids must be taken into account. Assume that the temperature of each body is uniform.
15. The same factors must be taken into account as in the discussion of Problem 14. Use combined convection-radiation coefficients.
16. Assume that the temperature throughout a perpendicular cross section of the rod is uniform. As boundary conditions we have that the temperatures of the ends of the rod are the same as those of the corresponding liquids. Otherwise the problem is like Problem 2.
17. We suppose that the plates are in a vacuum so that convection can be neglected. Assume uniform temperatures in the plates.
18. Assume axial symmetry for the bulb. Take into account storage of heat in the bulb, heat transfer to the bulb from the bath, heat conduction along the bulb, and loss of heat to the atmosphere by convection (and radiation). Assume that the temperature throughout a perpendicular cross section of the bulb is a constant.
19. This is a problem in heat storage in the bulb. Assume that the temperature throughout the bulb is constant. Take into account convection (and radiation).
20. Consider a steady state. In addition to heat balance equations, use also an equation for the flow of steam through the nozzle, and an equation stating that the mass rate of flow of steam to the nozzle equals the mass rate of loss of steam from the boiler. If the hollow tube is assumed to be long, an equation of flow must be written for it.
21. Suppose that the bulb is a point source of heat giving off heat energy at a certain rate.
22. Suppose that the water in the jacket is at a practically constant temperature. Heat goes to or from the cylinder wall by convection (and radiation), being lost (or gained) by the gas which may be expanding or contracting. Take heat storage in the wall into account. Assume that at each instant the gas and wall are at uniform temperatures.
23. Assume that the cylinder wall is at a uniform temperature, but take conduction in the fins into account.
24. Assume that the walls insulate the room perfectly, and that the temperatures of the inside air and also of the glass are uniform. Consider that the inside air gets heat by convection and radiation from the glass.
25. Suppose that the water and iron are each at a uniform temperature. Take into account storage of heat in the iron. Consider that the iron loses heat energy by radiation and convection.

26. Treat the combined convection and radiation to the air from the pipe. Assume that the temperature of the water throughout a perpendicular cross section of the pipe is a constant, and that this is true of the pipe wall. We have incompressible flow.
27. Assume that the temperature of the gas is uniform, and that this is also true of the wall of the sphere, and of the hot bath, the temperature of the latter being assumed constant. Take into account storage, convection, and radiation.
28. Let $T$ be the absolute temperature of the resistor and $T_a$ the absolute temperature of the air. Let $i$ be the current through the resistor, and $R$ the resistance. Let $w$ be the specific weight of the resistor element, $c_p$ its specific heat, $V$ its volume, and $h$ a coefficient of heat transfer by combined radiation and convection to the air. We have an energy equation of the form

$$Ri^2 = hA(T - T_a) + wc_pV\frac{dT}{dt}.$$

Neglect the variation of the resistance of the resistor with temperature. If $e$ is the applied voltage we have

$$e = Ri.$$

29. See the reference to Stodola at the end of Chapter 6.
30. See the reference to Zemansky at the end of Chapter 6.
31. See Section 6.15 concerning the 4-cycle diesel. For a 2-cycle diesel, exhaust begins early during the power stroke, and scavenging is accomplished by blowing air into the cylinder.
32. Suppose that a layer of coal at the surface burns, producing heat at a certain rate, that some heat is conducted to the interior, and the rest is lost to the exterior air by radiation and convection. Take the temperature gradient in the coal into account. Assume a perfectly symmetrical pile.
33. See the reference to McAdams at the end of Chapter 8.
34. See the same McAdams reference mentioned above.
35. Let $q$ be the rate of heat transfer, $D$ a typical linear dimension of the body, $V$ the speed of the fluid, $\Delta\theta$ the temperature difference between the body and the fluid, $c_p$ the specific heat of the fluid based on a unit volume [(drop $w$ in (8.3)] and $k$ the thermal conductivity of the fluid. Then, for a function $f$,

$$\frac{Dk\Delta\theta}{q} = f\left(\frac{Dc_pV}{k}\right).$$

36. Let the primary quantities be force $F$, mass $M$, length $L$, and time $T$, and use the dimensional constant $g_c$ for $ma/F$ of Newton's second law of motion. Then

$$\frac{g_cD}{\rho V^2}\frac{dp}{dl} = f\left(\frac{\mu}{DV\rho}\right),$$

where $D$ is the diameter of the pipe, $\rho$ is the density and $\mu$ the viscosity of the fluid, $V$ is the speed of the fluid, and $f$ denotes a function.
37. The secondary quantities are the conductivity $k$ and radius $r$ of the cylinder, the rate $q$ of heat generated per unit volume, and $\Delta\theta$. Take as primary quan-

tities $L$, $\theta$, $T$, and $B$, where $B$ is Btu per hour, and $L$ and $T$ are length and time. The solution is

$$\Delta\theta = C\frac{qr^2}{k}$$

for a constant $C$.

## PART IV

1. We have only tension in the cable tangent to the cable. The horizontal component of tension is a constant, denoted by $H$. Let $y$ be a vertical Cartesian coordinate, $w$ the weight per unit length, $x$ a horizontal coordinate, and $s$ arc length along the cable. In this case

$$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{w}{H}.$$

See the reference to Synge and Griffith at the end of Chapter 1.
2. Suppose that the rope is in the form of straight-line segments subject to tension only.
3. Let an $xy$-coordinate system be set up with $x$ along the beam, and $y$ pointed upward. Imagine that the beam rests on springs. We now have an equation of the form

$$EI\frac{d^4y}{dx^4} = p(x) - h(y)$$

where $p(x)$ is the load term. See the reference to Kármán and Biot at the end of Chapter 2.
4. Treat the blade as a beam.
5. Treat the wing as a beam.
6. The wire is subject to tension only. Take the shape of the wire at the instant the hammer leaves the wire as known, and assume a given velocity at each point. For the treatment of vibrating wires (or strings) see the reference to Kármán and Biot mentioned above.
7. Consider the beam to be one-dimensional with tension in the direction of the length of the beam, a shear force perpendicular to the tension, and a bending moment $M$ at each section. Assume continuity of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the supports. Write down a differential equation for each beam portion between supports. The continuity assumptions yield some initial conditions. See the reference to Synge and Griffith at the end of Chapter 1.
8. This is a plane problem. Make use of symmetry about the center $O$. See the reference to Prescott at the end of Chapter 10.
9. Apply the theory of beams bent by couples at the ends. Use the principle of superposition. See the reference to Love at the end of Chapter 9.
10. The axle is a beam with loads at two points. To simplify the discussion suppose that the wheels are rigid and vertical.

11. Let $\theta$ be the angular displacement of the flywheel. Then an equation of the form

$$I\theta'' + k\theta = 0$$

holds.

12. Consider only tension and thrust along the length of the beam. See the reference to Synge and Griffith at the end of Chapter 1.
13. This is a one-dimensional problem. The friction force inside between the pipe and the water is a function of the speed of the water. This force is balanced by longitudinal stress in the pipe.
14. Consider the strains due to the bending moments separately from the strains due to compression, and superimpose the results.
15. The track has loads at 12 points corresponding to the 12 wheels. Consider the rails to be beams loaded at these points, and resting on an elastic foundation.
16. Assume an initial position for the beam, and an initial velocity at each point.
17. The cable is under tension only. The beam is hinged at one end, and is loaded at the other.
18. See the sections on beams and cables in Part IV of this book.
19. Take the sphere under uniform pressure so that there is complete symmetry with respect to the center. See the reference to Prescott at the end of Chapter 10.
20. Write down equations for each half of the rod.
21. Make use of symmetry with respect to the center. See the reference to Prescott above.
22. Treat each half of the rod separately. Make use of the continuity of the tension force at the junction.
23. Treat the rod halves separately.
24. Assume uniform twist. See the reference to Prescott above.
25. The cable applies forces at the pole ends. These poles are beams, whence beam theory can be applied.
26. Let $\theta$ be the angular displacement of the weight from equilibrium. Obtain an equation of the form

$$I\theta'' + k\theta = 0.$$

27. Consider the bridge to be a cable, and a beam supported from the cable by hangers. Assume that the cable carries its own weight, the weight of the hangers, and the dead weight of the beam without producing a bending moment in the beam. Neglect elastic deformation of the hangars. Assume that the moment of inertia $I$ of the cross section of the beam is a constant. Obtain an equation of the form

$$EI\frac{d^4y}{dx^4} - (H + h)\frac{d^2y}{dx^2} = p - q\frac{h}{H},$$

where $E$ is Young's modulus, $y$ is the vertical deflection of a point on the cable (or beam), and $H$ is the horizontal tension in the cable for dead weight; additional tension produced by live weight is $h$, dead load per unit length is $q$, and line load applied to the beam (truss) is $p$ per unit length; $x$ is the horizontal coordinate. See the reference to Kármán and Biot at the end of Chapter 2.

28. We have symmetry about the axis of the rod. Consider 2-dimensional stress and strain in a cross section of the rod perpendicular to its axis.
29. See the reference to Prescott at the end of Chapter 10.
30. See the same Prescott reference mentioned above.
31. See the same Prescott reference mentioned above.
32. The secondary quantities are speed $V$, specific weight $w$, modulus of elasticity $E$, and the magnitude $g$ of the acceleration of gravity. The primary quantities are mass $M$, length $L$, and time $T$. The dimensions of the quantities $V$, $w$, $E$, and $g$ are $L/T$, $ML^{-2}T^{-2}$, $ML^{-1}T^{-2}$, and $LT^{-2}$ respectively. There is one dimensionless term with $V$ to the first power, namely $Vw^{\frac{1}{2}}E^{-\frac{1}{2}}g^{-\frac{1}{2}}$, whence

$$V = K\sqrt{\frac{Eg}{w}}$$

for a constant $K$.

33. The secondary quantities are the stiffness $S$, length $l$, breadth $b$, depth $d$, modulus of elasticity $E$, and shear modulus $G$ of the beam. The dimensions of $S$, $E$, and $G$ are $MT^{-2}$, $ML^{-1}T^{-2}$, and $ML^{-1}T^{-2}$ respectively. The solution is

$$S = Elf\left(\frac{b}{l}, \frac{d}{l}, \frac{G}{E}\right)$$

for a function $f$.

34. The secondary quantities are the specific weight $W$, displacement $w$, modulus of elasticity $E$, length $l$ of the bar, and distance $z$ from the top of the bar to the point of displacement. The desired relation connecting three $\pi$-terms is

$$\frac{w}{l} = f\left(\frac{z}{l}, \frac{Wl}{E}\right),$$

where $f$ denotes a function.

## PART V

1. Assume a steady state, and that the speed of the water is undisturbed by contact with the plate, so that if the water has speed $V$ before contact, each part will have speed $V$ after contact, but a different direction. Suppose that the water divides into two parts, one flowing along the plate in the onward direction at a volumetric rate $Q_1$, and the other in the opposite direction at a volumetric rate $Q_2$. Let $Q$ be the volumetric rate of the incoming water, and $w$ the specific weight. Suppose that the plate is perpendicular to the plane of the paper in Figure 203 and that its trace on this plane makes an angle $\theta$ with the positive half of the $x$-axis. From the continuity of flow and the fact that there is no force parallel to the plate (using momentum considerations) we obtain

$$Q_1 = \tfrac{1}{2}Q(1 + \cos\theta),$$
$$Q_2 = \tfrac{1}{2}Q(1 - \cos\theta).$$

Since the component of force in the $x$-direction is equal to the rate of change of momentum, we have

$$\frac{QwV}{g} - \left(\frac{Q_1wV}{g} - \frac{Q_2wV}{g}\right)\cos\theta = kx$$

for the spring force $kx$ and the magnitude $g$ of the acceleration of gravity. See pp. 114 and 115 of the reference of Dodge and Thompson at the end of Chapter 12.

2. Add a term of the form $mx''$ to $kx$ in the solution of Problem 1, and take $\theta$ equal to 85°. As initial conditions we have

$$t = 0,\ x = 0,\ x' = 10.$$

3. The torque on the wheel is of the form

$$\frac{W}{g}(r_1V_{u1} - r_2V_{u2}),$$

where $W$ is the weight of water flow per second to the bucket, $g$ is the magnitude of the acceleration of gravity, $r_1$ is the radius of the wheel at which the incoming water strikes the wheel, $r_2$ is the radius of the wheel where the water leaves the bucket, $V_{u1}$ is the component of the speed of the incoming water where this component is tangential to the circle described by the part of the bucket that the water strikes, and $V_{u2}$ is the component of the speed of the outgoing water tangential to the circle described by the part of the bucket that the water leaves. See R. L. Daugherty, Hydraulic Turbines, McGraw-Hill, New York, 1920, pp. 87–90.

4. The equation (12.41) is replaced by

$$F + mz'' = (p - p_a)A - k_2z$$

for the friction force denoted by $F$. The equation (12.36) is replaced by

$$A_1(p - p_a) + \frac{dz}{dt} = Kx\sqrt{p_0 - p},$$

where the first term represents leakage.

5. See Section 12.11. The equation

$$-10^3\frac{dx}{dt} = 8\sqrt{x} + 4\sqrt{x - 10},$$

and the initial condition

$$t = 0,\ x = 30$$

hold.

6. See pp. 51–53 of the book by Russell referred to at the end of Chapter 12.
7. Merely modify the equations of Section 13.13 to take care of friction and leakage. Assume that the friction is constant for the piston under motion, and treat the leakage as if it were proportional to the pressure differential across the piston.
8. See pp. 62–66 of the book by Dodge and Thompson referred to at the end of Chapter 12.

9. See the reference above to Dodge and Thompson.
10. Suppose that the pipes are horizontal. Use water hammer theory.
11. Assume that $u$ varies linearly with respect to time at $x = 2400$, that is,

$$u = -120\frac{t}{T}$$

for the time of closure $T$.
12. Assume that the tension on the surface is uniform in all directions and the same at all points. The general gas law $pV = nRT$ applies to the volume $V$, pressure $p$, and absolute temperature $T$ of the enclosed gas. The bouyant force is due to the difference in pressures above and below and is the weight of the displaced air, assuming that the weight of the hydrogen can be neglected. An equation of the form

$$mx'' = \rho g V$$

holds for the vertical coordinate $x$ of the balloon. From the tension on the surface of the balloon a relation between $V$ and $p$ can be obtained. Assume that $T$ is a known function of $x$. Adjoin to the equations the initial conditions

$$t = 0, x = x' = 0.$$

13. Use the orifice flow law, the general gas law for the gas inside the balloon and the continuity gas equation, namely, that the rate of mass flow of the gas out of the balloon is equal to the rate of decrease of the mass of the gas in the balloon.
14. Let the low pressure be denoted by $p_1$, the high pressure by $p_2$, the height of the surface of the low pressure oil above the pump by $x$, the corresponding height for the high pressure oil by $y$, and the pump speed by $N$. The pressures ahead and behind the pump are of the form $(p_1 + wx)$ and $(p_2 + wy)$, respectively, and the rate of oil flow through the pump is a function $f[N, (p_2 + wy) - (p_1 + wx)]$. Take the leakage proportional to the pressure differential across the pump. Because the total volume of oil in the system is a constant there is a known relation between $x$ and $y$.
15. The tank is a reservoir. Apply water hammer theory.
16. See the remarks above on Problem 15.
17. From the pressure on the oil surface we can compute the pressure just ahead of the pipe. Write down a differential equation for the pipe, and one for the orifice.
18. Assume that the water travels with a plane front perpendicular or almost perpendicular to the axis of the pipe. Let the front have a pipe coordinate $x$. Let $V$ be the volume of the air, and $p$ its pressure and suppose that a polytropic relation

$$pV^\gamma = C$$

holds. Suppose that the pipe has a constant cross section area $A$, and that the $x$-coordinate of the water front increases with time. We now have

$$\frac{dV}{dt} = -A\frac{dx}{dt}.$$

Let $p_0$ be the pressure of the water entering the pipe. We may suppose that the rate of water flow by volume into the pipe is proportional to $(p_0 - p)/x$, provided that the origin of the $x$-axis is taken at the start of the column of water in the pipe. Then

$$A\frac{dx}{dt} = \frac{k(p_0 - p)}{x}$$

for a constant $k$.

19. Let $p_a$ be the absolute pressure of the atmosphere, let $x$ be the height of the water in the first tank which originally contains water, and $y$ the height of the water in the other tank, these heights being measured from the orifice. Suppose that $A_1$ and $A_2$ are the horizontal cross section areas of the first and second tanks respectively, where these tanks are assumed to be of uniform cross section, and let the effective area $A$ of the orifice satisfy the relation

$$A = A_0 \sin \omega t$$

for constants $A_0$ and $\omega$. Neglecting the water in the pipe, the relation

$$xA_1 + yA_2 = k,$$

for a constant $k$, expresses the fact that the total volume of water is a constant. The speed $V$ of the water through the orifice is given by

$$V = \sqrt{\frac{2g}{w}[(p_a + wx) - (p_a + wy)]},$$

where $w$ is the specific weight of the water, and $g$ is the magnitude of the acceleration of gravity. The volumetric rate at which the level in the first tank drops is equal to the rate of flow of the water through the orifice, so that

$$AV = -A_1\frac{dx}{dt}.$$

We need the initial condition

$$t = 0,\ x = x_0$$

for a constant $x_0$.

20. Suppose that the oil has a front which is in a plane perpendicular to the cylinder axis. Assume that the oil supply pressure $p_0$ is high enough so that we can neglect the weight of the oil, and suppose that atmospheric pressure $p_a$ applies to one side of the piston. Let $p$, $V$, and $T$ be the pressure, volume, and temperature of the confined air. For this air a relation of the type

$$pV = nRT$$

holds, where $n$ is the number of moles of the air. Let $w_a$ be the specific weight of the air. The speed $v$ of the air leaking past the piston, assuming that the air travels through a space which can be treated as an orifice, satisfies

$$v = \sqrt{\frac{2g}{w_a}(p - p_a)}.$$

We can write down an equation stating that the mass rate of flow of the air past the piston is equal to the mass rate of loss of air from the cylinder. The volumetric rate of flow of the oil is proportional to $\sqrt{p_0 - p}$. This is equal to

$$A\frac{dx}{dt},$$

where $x$ is the coordinate of the oil front relative to a Cartesian $x$-axis parallel to the cylinder elements, and $A$ is the cross section area of the interior of the cylinder. Also

$$\frac{dV}{dt} = A\left(\frac{dy}{dt} - \frac{dx}{dt}\right),$$

where $y$ is the $x$-coordinate of the piston. Let $m$ be the mass of the piston, and $k$ the scale of the spring. Now

$$my'' + ky = A(p - p_a).$$

We can thus obtain five equations in the five variables $x$, $y$, $V$, $p$, and $n$, the temperature $T$ being assumed constant; or if not, suppose that a polytropic relation

$$pV^\gamma = C$$

holds.

**21.** Let $x$ be the vertical coordinate of the submarine, and $m$ its mass. Write an equation stating that $mx''$ equals the weight of the water displaced by the submarine. Use the initial conditions

$$t = 0,\ x = x' = 0.$$

**22.** Let $m$ and $x$ be the mass and vertical coordinate of the body, respectively. Write an equation stating that $mx''$ equals the weight of the displaced water. Write another equation stating that the weight of the water displaced varies as a certain function of $x$. Adjoin initial conditions.

**23.** The treatment of this problem is similar to that of Problem 12.

**24.** It is convenient to use two-dimensional potential flow here.

**25.** Assume symmetry about a vertical axis. Use partial differential equations. The pressure of the atmosphere is the same at all points on the surface of the water.

**26.** See the discussion of surface tension in pp. 6–8 of the Russell reference mentioned at the end of Chapter 12.

**27.** See the reference to Liepmann and Puckett at the end of Chapter 14.

**28.** The secondary quantities are $l$, $V$, $dp/dx$, and $\mu$ (for these notations see Section 12.23). The solution is

$$\frac{l^2\dfrac{dp}{dx}}{V\mu} = K$$

for a constant $K$.

**29.** Let the roughness of the pipe be described by a linear parameter $k$. The solution is

$$\frac{dp}{dx} = \frac{\rho V^2}{l} f\left(\frac{Vl\rho}{\mu}, \frac{l}{k}\right)$$

for a function $f$.

**30.** The secondary quantities are a characteristic area $S$, the drag $D$, limiting speed $V$ of the air as we recede from the airplane, and the density $\rho$ of the air. There is one dimensionless product with $D$ to the first power, namely $D\rho^{-1}V^{-2}S^{-1}$, whence

$$D = K(\rho V^2)S$$

for a constant $K$. This throws more light on the formula for drag in Section 14.3.

**31.** The secondary quantities are the speed $V$ of the wave, the density $\rho$ of the water, the magnitude $g$ of the acceleration of gravity, and the wave length $\lambda$. The solution is

$$V = C\sqrt{\lambda g}$$

for a constant $C$.

**32.** Let the primary quantities be force $F$, length $L$, and time $T$. The secondary quantities are the speed $V$ of the sound, the density $\rho$, pressure $p$, and viscosity $\mu$ of the air, with the dimensions, $LT^{-1}$, $FT^2L^{-4}$, $FL^{-2}$, and $FTL^{-2}$ respectively. Here $V\rho^{\frac{1}{2}}p^{-\frac{1}{2}}$ is dimensionless and

$$V = C\sqrt{\frac{p}{\rho}}$$

is the desired formula, $C$ being a constant.

**33.** The secondary quantities are $V$, the magnitude $g$ of the acceleration of gravity, the pressure head $H$, the viscosity $\mu$ and density $\rho$ of the fluid. The quantities

$$\frac{VH\rho}{\mu}, \frac{gH^3\rho^2}{\mu^2}$$

are dimensionless. By the Buckingham pi theorem

$$V = \frac{\mu}{H\rho} f\left(\frac{gH^3\rho^2}{\mu^2}\right)$$

for a function $f$. Since $V$ is proportional to $H^{\frac{1}{2}}$

$$V = C\sqrt{gH},$$

where $C$ is constant. Thus $V$ is independent of $\mu$ and $\rho$.

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